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THE PROPAGATION OF PLANE ACOUSTIC WAVES IN A RADIATING GAS

By BARRETT STONE BALDWIN, Jr.

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59

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CONTENTS

ILLUSTRATIONS

TECHNICAL REPORT R-138

THE PROPAGATION OF PLANE ACOUSTIC WAVES IN A RADIATING GAS

By Barrett Stone Baldwin, Jr.

SUMMARY

A study is made of the interaction of thermal radiation and fluid flow in the acoustic approximation. An earlier investigation is reviewed wherein the disturbances produced in a semi-infinite expanse of radiating gas by sinusoidal motion or temperature variation of a plane radiating wall are analyzed. The one-dimensional unsteady-flow equations applicable to this problem are generalized to include the effects of a frequency-dependent radiation-absorption coefficient. It is found that a single integro-differential equation of the same form as that previously given is obtained. It is demonstrated that by a redefinition of parameters, a previously given solution applies for a frequency-dependent absorption coefficient as well as for a grey gas. The solution appears, in general, as the sum of two types of harmonic traveling waves: (1) an essentially classical sound wave, but with slightly altered speed and a small amount of damping, and (2) a radiation-induced wave whose speed and damping may be either large or small, depending on the frequency of oscillation and the condition of the gas.

The previously given solution for sinusoidal motion of the wall is used in superposition to find the response to an impulsively moving wall. The general results are given in the form of integral expressions for the linearized disturbance quantities. A closed-form approximation for the velocity disturbance is obtained. It is found that the main part of the response to an impulsive motion of the wall propagates initially at the isentropic speed of sound. As it progresses, however, the wave becomes dispersed and its main part travels at a somewhat slower speed. Eventually, the main part of the disturbance shifts back to the isentropic speed. Some components of the response, associated with the radiation-induced wave system, travel at speeds up to the velocity of light. As a result, there is a small precursor to the main wave front. This part of the response dies out exponentially with distance ahead of the main disturbance.

The validity of several approximations used in the derivations is investigated. The second-order equations are derived to help establish the conditions under which the linearized results are a first approximation to the original nonlinear equations. An integro-exponential function appearing in the basic equations was approximated by an exponential throughout the present work. This expedient was introduced in early solutions of astrophysical problems. The validity of this approach for the present problem is investigated by means of a two-term approximation and by considering the properties of an exact solution for a grey gas. It is found that the higher derivatives of the flow quantities are not given correctly by the approximation in a small region near the wall. However, valid results are obtained for the flow quantities themselves everywhere.

The integral expressions representing the response to an impulsive motion of the wall are evaluated exactly in closed form for a number of special cases. This is done to provide information on the pressure and temperature fields and as a check on the approximate evaluation of the velocity disturbance. Additional checks are made by means of numerical evaluations of the integral expression for the velocity.

Finally, the problem of the response to a sudden change in the temperature of a fixed wall is considered briefly. In this case, a small velocity disturbance builds up and then decays to zero. Near the wall, the gas temperature relaxes exponentially from its original value to that of the wall. At large distances from the wall, the temperature variation is of a type characterized by a diffusion process.

INTRODUCTION

The present work is an extension of the investigation reported in reference 1 concerning the interaction between thermal radiation and fluid flow in the acoustic approximation. This field of study was initiated in 1851 by Stokes (see ref. 2). His purpose was to show that thermal radiation does not affect the propagation of sound under ordinary conditions. This accomplished, such an interaction was not further considered until recently. Since 1956 a number of papers on the propagation of weak disturbances in a radiating gas have appeared.

Stokes' investigation was based on an approximation appropriate to highly transparent, low-temperature air. Heat exchange was assumed to take place between each element of gas and a reservoir at the temperature of the undisturbed gas. (For a brief outline of this work see also ref. 3.) A parameter that is a measure of the rate of heat transfer at a given temperature difference has been called the Stokes coefficient by later authors. The acoustic equation resulting from the inclusion of such a process is a third-order partial differential equation. In the past decade two comprehensive surveys on the propagation of sound in fluids have appeared in the literature (see refs. 4 and 5). In these the treatment of the effects of radiative heat transfer is based on the Stokes approximation. In reference 6 a method for evaluating the Stokes coefficient is developed which yields useful information. This work utilizes the correct integral expression for the radiative heat transfer between elements of the gas. The solution is assumed to be of the same form as that resulting from the Stokes approximation and this leads to an evaluation of the Stokes coefficient. In addition acoustic wave speeds and damping constants are found which compare favorably with the results from more recent investigations.

So far as the author is aware, the first complete theory for the effect of radiative heat transfer on the propagation of sound far from any obstacle was developed by V. A. Prokof'ev (see ref. 7). Earlier work in this field by Prokof'ev is referred to in reference 7. This author considers the problem of thermally radiating acoustic waves in great generality, including the effects of viscosity and thermal conductivity as well as the smaller effects (for aerodynamic purposes, at least) of radiation scattering, radiation pressure, and the direct contribution of radiation to internal energy.

In all of the foregoing works, except reference 1, attention is confined to the propagation of sound waves in the gas far from any boundaries. In reference 1 an infinite, plane radiating wall is introduced and taken to be the source of the disturbances. Following a pattern established in previous investigations of chemical and vibrational relaxation effects, the influence of the nonequilibrium process under study is isolated by neglecting other complications. In references 8–11, for example, a single chemical or vibrational nonequilibrium process is introduced into the governing equations with all other processes taken to be in equilibrium. Also, in these works attention is confined to one-dimensional unsteady or two-dimensional steady flows with simple boundary conditions. Reference 1 and the present work, taken together, represent an attempt to include flows with radiative heat transfer in this category of flow fields involving a single nonequilibrium process. In the case of chemical and vibrational nonequilibrium, it was found that the same governing differential equation applies for either process in the small-disturbance approximation. It will be seen that the nonequilibrium effect due to radiation does not fall in the same class, although there are certain similarities.

The present investigation follows reference 1 in assuming that nonequilibrium effects from processes such as molecular transport, dissociation, vibration, etc., are negligible. Radiation scattering, radiation pressure, and the contribution of radiation to internal energy are also neglected. For simplicity, the gas is assumed to be perfect. The radiative effects are taken into account on the basis of the usual quasi-equilibrium hypothesis, wherein a Boltzmann distribution of excited states is assumed. The geometrical configuration to be considered is that of a semi-infinite expanse of radiating gas on one side of an infinite, plane radiating wall. Initially the gas is assumed to be at rest, in a uniform state, and at a temperature equal to that of the wall. One-dimensional disturbances can then be produced in the gas by moving the wall at constant temperature, or by varying the temperature of a fixed wall, or both.

In its one-dimensional character, the present problem is related to the classical astrophysical problem of the plane-parallel stellar atmosphere (see, for example, ref. 12 or 13). For that case, however, the fluid motion is negligible and there is no wall. The treatment of radiation in the plane-parallel case has recently been extended to include the effects of fluid motion and solid boundaries (see refs. 14, 15, and 16). In reference 1 the wall boundary condition is generalized, and the basic equations are used to derive an integro-differential acoustic equation for a grey gas (absorption coefficient independent of optical frequency). This analysis is generalized slightly in Section I of the present work by including the effect of a frequency-dependent absorption coefficient. The resulting linear integro-differential equation is of the same form as that which applies for a grey gas. The only difference is that an integro-exponential function, appearing as an attenuation factor in the equation for a grey gas, is replaced by a more complicated function involving an integral over optical frequency. It is found that the radiative properties of the gas enter the equation only in their effect on the form of the attenuation factor and the value of a mean absorption coefficient, both evaluated at the undisturbed condition of the gas. To help establish that the present linearization is imbedded in a successive approximation procedure, the second-order equations are derived in appendix B.

In reference 1, with the aid of a suitable approximation to the attenuation factor appearing in the radiation terms, the acoustic equation for a grey gas is solved for the case of a black wall undergoing sinusoidal variations. A similar procedure in Section II of the present work leads to a solution of the same form for a nongrey gas. This approximation, wherein an integro-exponential function is approximated by an exponential, was first used in an early solution of the stellar-atmosphere problem (see ref. 12). The validity of the approximation for the problem considered here is investigated in appendices C and D. Appendix D contains discussion of a procedure for finding an exact solution for a grey gas. It is found that in the exact solution the higher derivatives of the flow quantities must be singular at the wall. The approximate results do not reproduce this effect; however, the flow quantities themselves are adequately approximated everywhere.

A large fraction of the present effort goes into finding the response of the gas to an impulsive motion of a wall at fixed temperature. In section III, the solution for this case is found by superposition of the sinusoidal solutions using Fourier-transform theory. The general results are given in the form of integral expressions for the linearized disturbance quantities. These are evaluated exactly in closed form, for certain limiting values of the variables, in appendix E. A closed-form approximation for the velocity disturbance at all values of the variables is derived in appendix F, and this is checked by numerical evaluations in appendix G.

The results from the entire investigation of the response to an impulsive motion of a wall at fixed temperature are summarized and discussed in Section IV. It is found that the main part of the resulting wave propagates initially at the isentropic speed of sound. As it progresses, the wave becomes dispersed and travels at a slower speed, which, for a high gas temperature, becomes the isothermal speed proposed by Newton for sound waves. These findings are qualitatively similar to those for a gas in chemical or vibrational nonequilibrium. (For that case, the initial velocity is the frozen speed of sound, and the final propagation velocity is that corresponding to equilibrium.) After reaching the slower speed, however, the subsequent behavior of the compression wave in the radiating gas differs from that associated with the other nonequilibrium processes. In particular, the main part of the disturbance eventually shifts back to the higher isentropic speed. In addition, some components of the response travel at speeds up to the velocity of light. As a result, there is a small precursor to the main wave front at all of its positions. This part of the response dies out exponentially with distance ahead of the main disturbance.

Finally, the problem of the response to a step variation in the temperature of a fixed wall is formulated and carried to partial completion in appendix H. In this case, a small velocity disturbance builds up and then decays to zero. Near the wall, the temperature relaxes exponentially from its initial value to a final value equal to that of the wall. Far from the wall, the temperature variation is of a type characterized by a diffusion process.

In the present work, emphasis is placed on fluid-dynamical effects rather than on the physics that goes into the determination of the absorption coefficient. When evaluation of the absorption coefficient is considered, however, common ground is established with another class of aerodynamic problems that have received increased attention recently. In these problems the motivating consideration is the evaluation of the heat transfer to a space vehicle during reentry into the earth's atmosphere. For this purpose, early theoretical and experimental work on the radiative properties of air was carried out at the Avco Everett Research Laboratory (see ref. 17 and refs. thereof). A useful semiempirical theory for the emissivity of hot air is also given in reference 18. To assess the contribution of radiation to reentry heat transfer, the effect of the radiation on the fluid motion was at first neglected (see ref. 19). In reference 20, however, a simplified problem is solved wherein the interaction between the radiation and fluid flow is taken into account. This work includes information on other high-temperature properties of air, which would be of interest in any attempt to compare the results of the present work with experiment.

A related problem, concerning the effect of radiation on shock-wave structure, is considered in reference 21. It is found that, for weak waves in a gas at high temperature, the structure of the shock may be determined by radiative heat transfer, with neligible effects from viscosity and thermal heat conduction. It is a matter of future interest to investigate the relationship of this result to the linearized solution for weak waves found in the present work.

Before concluding the discussion of gas properties, three other pertinent investigations should be mentioned. Reference 22 contains an excellent review for aerodynamicists of the basic ideas involved in the theory of radiative heat transfer, including an application of quantum-mechanical theory in the evaluation of the radiative properties of oxygen. Reference 23 contains a similar analysis for a monatomic gas, and the results are applied in a study of shock-tube flow. Finally, the results of the present work may find application in investigations of very low frequency waves in the atmosphere. For information on the radiative properties of air under atmospheric conditions, see reference 24.

The small-disturbance inviscid-flow theory that has evolved over the years takes account of the effects of viscosity and thermal heat conduction by replacing shock waves and boundary layers with discontinuities. The same procedure is used in the present work. Thus the gas immediately adjacent to the wall, when disturbed, may arrive at a temperature different from that of the wall by virtue of the presence of an optically thin thermal boundary layer. For supersonic or unsteady flow, cumulative nonlinear effects appear at large distances from the source of a disturbance even in the lowest-order small-disturbance approximation. Such effects can be taken into account by a straining of the coordinate system (see refs. 25 and 26). Presumably the coordinate-stretching process can be carried out here, but the matter will not be investigated at this time.

I. ACOUSTIC EQUATIONS FOR A RADIATING GAS

The linearized equations for one-dimensional unsteady flow of a radiating gas are given in reference 1. The coordinate system, showing the x axis to be perpendicular to the bounding infinite plane wall, is depicted in figure 1. The semi-infinite expanse of radiating gas lies in the direction of the positive x axis from the wall. The wall is at $x=x_w(t)$, allowance being made for its motion. In the present work, the gas will not be assumed grey as in reference 1, but the wall will be considered black. In that case the acoustic equation of reference 1 prior to the inclusion of the grey-gas assumption is

$$\frac{\partial^2\varphi}{\partial t^2}-a_0^2\frac{\partial^2\varphi}{\partial x^2}=-\frac{(\gamma-1)}{\rho_0}Q \qquad (1)$$

where φ is a potential in terms of which the perturbation velocity, pressure, temperature, and density are given by the relations

$$u'=\frac{\partial\varphi}{\partial x} \qquad (2)$$

$$p'=-\rho_0\frac{\partial\varphi}{\partial t} \qquad (3)$$

$$\frac{\partial T'}{\partial t}=-\frac{1}{R}\left(\frac{\partial^2\varphi}{\partial t^2}-\frac{a_0^2}{\gamma}\frac{\partial^2\varphi}{\partial x^2}\right) \qquad (4)$$

$$\frac{\partial\rho'}{\partial t}=-\rho_0\frac{\partial^2\varphi}{\partial x^2} \qquad (5)$$

The prime is used to denote a perturbation quantity. The subscript 0 denotes the condition of the undisturbed gas wherein the gas is at rest, in a uniform state, and at a temperature equal to that of the wall. For example, ρ_0 is the gas density in the undisturbed state, and ρ' is equal to $\rho-\rho_0$. The symbols a_0, γ, and R have the conventional meanings so that a_0 represents the isentropic speed of sound in the undisturbed gas, γ the ratio of specific heats for a perfect gas, and R the gas constant. All symbols are defined in appendix A. A derivation of equations (1)–(5) is contained in appendix B.

The quantity Q appearing in equation (1) is the net radiant energy absorbed by the gas per unit volume and time. In general, Q can be analyzed in terms of its components Q_ν for particular optical frequencies ν, where

$$Q=\int_0^\infty Q_\nu d\nu \qquad (6)$$

An expression for Q_ν is derived in reference 1 for quite general properties of the wall and the gas. The subsequent derivation there is specialized to the case of a grey gas. In the generalization to the case of a nongrey gas, to be treated here, the wall will be considered black. The appropriate expression for Q_ν can be obtained by setting the quantity ϵ_ν equal to one in equations (20) and (21) of reference 1. The result is

$$Q_\nu=2\pi\alpha_\nu\Big[B_\nu(T_w)E_2(\eta_\nu)+\int_0^{\eta_\nu}B_\nu(\tilde{T})E_1(\eta_\nu-\tilde{\eta}_\nu)d\tilde{\eta}_\nu +\int_{\eta_\nu}^\infty B_\nu(\tilde{T})E_1(\tilde{\eta}_\nu-\eta_\nu)d\tilde{\eta}_\nu-2B_\nu(T)\Big] \qquad (7)$$

Equations (6) and (7) have not yet been linearized. They will be rearranged and linearized to obtain an expression for Q in terms of φ for use in equation (1). Equation (7) applies in particular to the case of a black wall, T_w being the wall temperature. The quantity α_ν is the absorption coefficient of the gas at the radiation frequency ν

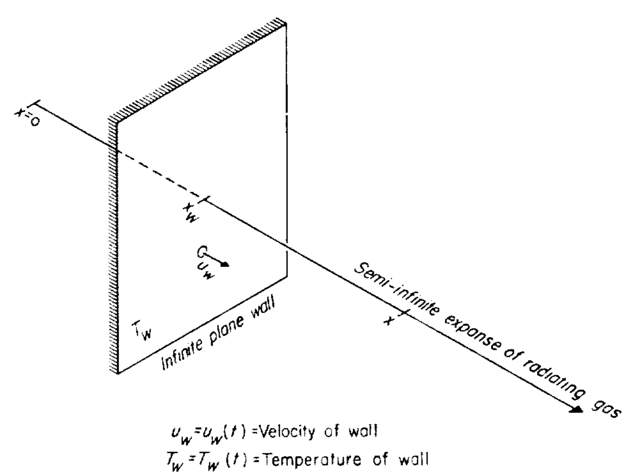


FIGURE 1.—Coordinate system.

as defined, for example, in reference 22. The function $B_\nu(T)$ is the Planck function

$$B_\nu(T)=\frac{2h\nu^3}{c^2}\,\frac{1}{e^{h\nu/kT}-1} \tag{8}$$

and $E_1(z)$, $E_2(z)$ are the integro-exponential functions

$$E_n(z)=\int_0^1 \mu^{n-2}e^{-z/\mu}d\mu, \qquad n=1,2 \tag{9}$$

For a discussion of the properties of these functions including a number of integral relations, see reference 27.

The quantity η_ν is the optical depth from the wall for radiation of frequency ν, and is given by

$$\eta_\nu=\int_{x_w(t)}^{x} \alpha_\nu(\hat{x})d\hat{x} \tag{10}$$

The temperatures T and $\tilde{T}$ that appear in equation (7) are the gas temperatures at the positions corresponding to η_ν and $\tilde{\eta}_\nu$, respectively.

A useful alternative form of equation (7) is the relation found from it through an integration by parts

$$Q_\nu=2\pi\alpha_\nu\left\{[B_\nu(T_w)-B_\nu(T)|_{\eta_\nu=0}]E_2(\eta_\nu)\right.$$
$$-\int_0^{\eta_\nu}\frac{dB_\nu(\tilde{T})}{d\tilde{T}}\,E_2(\eta_\nu-\tilde{\eta}_\nu)\,\frac{\partial\tilde{T}}{\partial\tilde{\eta}_\nu}\,d\tilde{\eta}_\nu$$
$$\left.+\int_{\eta_\nu}^{\infty}\frac{dB_\nu(\tilde{T})}{d\tilde{T}}\,E_2(\tilde{\eta}_\nu-\eta_\nu)\,\frac{\partial\tilde{T}}{\partial\tilde{\eta}_\nu}\,d\tilde{\eta}_\nu\right\} \tag{11}$$

It should be noted that the partial derivative $\partial T/\partial\tilde{\eta}_\nu$ is used to indicate that t is held fixed. A further rearrangement of this expression for Q_ν will be made such that upon substitution in equation (6), the integration over ν can be carried out.

In the grey-gas approximation (α_ν independent of ν), an average absorption coefficient α and a corresponding optical depth η are introduced to replace α_ν and η_ν. The method of averaging is arbitrary, depending on the weighting function. According to reference 16, the Planck mean absorption coefficient and optical depth are defined by the relations

$$\alpha=\frac{\int_0^\infty B_\nu(T)\alpha_\nu d\nu}{\int_0^\infty B_\nu(T)d\nu} \tag{12}$$

$$\eta=\int_{x_w(t)}^{x}\alpha(\hat{x})d\hat{x} \tag{13}$$

When α_ν and η_ν are replaced by α and η in equation (11), and the result is substituted in equation (6), the integration over ν can be carried out explicitly by an interchange in the order of integration. This is so because the only remaining ν dependence is in the quantity $B_\nu(T)$, given in equation (8).

The grey-gas approximation will not be used in the following derivation. In the analysis of a real gas, for which α_ν varies with frequency, the integration over ν cannot be dispensed with so readily. However, it will be seen that the averaged quantities α and η are still convenient variables for the more general case.

As a first step toward removing the ν dependence in equation (11), we note that, for fixed ν and t, there is a functional relationship between η and η_ν so that, by a transformation, η_ν can be replaced with η as the variable of integration. For this purpose, the following relations can be established by comparison of equations (10) and (13).

$$d\eta_\nu=\frac{\alpha_\nu}{\alpha}\,d\eta \qquad \text{(for fixed } \nu \text{ and } t) \tag{14}$$

$$\eta_\nu=\eta_\nu(\eta,\, t)=\int_0^\eta \frac{\alpha_\nu}{\alpha}\,d\hat{\eta} \tag{15}$$

It follows that

$$\frac{\partial\tilde{T}}{\partial\tilde{\eta}_\nu}\,d\tilde{\eta}_\nu=\frac{\partial\tilde{T}}{\partial\tilde{\eta}}\,d\tilde{\eta} \tag{16}$$

Equation (11) can then be written as

$$Q_\nu=2\pi\alpha_\nu\left\{[B_\nu(T_w)-B_\nu(T)|_{\eta=0}]E_2(\eta_\nu)\right.$$
$$-\int_0^{\eta}\frac{dB_\nu(\tilde{T})}{d\tilde{T}}\,E_2(\eta_\nu-\tilde{\eta}_\nu)\,\frac{\partial\tilde{T}}{\partial\tilde{\eta}}\,d\tilde{\eta}$$
$$\left.+\int_{\eta}^{\infty}\frac{dB_\nu(\tilde{T})}{d\tilde{T}}\,E_2(\tilde{\eta}_\nu-\eta_\nu)\,\frac{\partial\tilde{T}}{\partial\tilde{\eta}}\,d\tilde{\eta}\right\} \tag{17}$$

The integral over ν could now be completed except for the ν dependence in the arguments of the E_2 function. In fact, this can still be done in a small-disturbance approximation. This comes about because when equation (17) is linearized, the ν dependent factors no longer depend on the temperature variations in the flow field. Then the integration over ν can be carried out once and for all.

In the acoustic approximation Q, and hence Q_ν, is needed only to lowest order in the departure from the undisturbed gas condition. Prior to a disturbance, the gas is at a uniform temperature equal to that of the wall. It follows that $B_\nu(T_w)-B(T)|_{\eta=0}$ and $\dfrac{\partial T}{\partial \eta}$ are perturbation quantities. As a result, to lowest order, the other factors in each term of equation (17) can be evaluated at the undisturbed condition of the gas (subscript 0). To accomplish this, the following relationships can be established by power series expansion

$$B_\nu(T)=B_\nu(T_0)+\frac{dB_\nu(T_0)}{dT_0}\,T'+0(T'^2) \qquad (18)$$

$$B_\nu(T_w)=B_\nu(T_0)+\frac{dB_\nu(T_0)}{dT_0}\,T'_w+0(T'^2_w) \qquad (19)$$

where

$$T'=T-T_0,\; T'_w=T_w-T_0$$

Since T_0 is constant, differentiation leads to the relations

$$\frac{dB_\nu(T)}{dT}=\frac{dB_\nu(T)}{dT'}=\frac{dB_\nu(T_0)}{dT_0}+0(T')$$

$$\frac{\partial T}{\partial \eta}=\frac{\partial T'}{\partial \eta}$$

Combination of these relations yields

$$B_\nu(T_w)-B_\nu(T)|_{\eta=0}=\frac{dB_\nu(T_0)}{dT_0}\,(T'_w-T'|_{\eta=0})+0(T'^2)$$

$$\frac{dB_\nu(T)}{dT}\,\frac{\partial T}{\partial \eta}=\frac{dB_\nu(T_0)}{dT_0}\,\frac{\partial T'}{\partial \eta}+0(T'^2)$$

The symbol $0(T'^2)$ is intended to include second-order terms of the form $T'\,\dfrac{\partial T'}{\partial \eta}$ and T'^2_w. The last two expressions can be used to write equation (17) as

$$Q_\nu=2\pi\alpha_{\nu_0}\left[\frac{dB_\nu(T_0)}{dT_0}\,E_2(\eta_\nu)(T'_w-T'|_{\eta=0}) -\int_0^\eta \frac{dB_\nu(T_0)}{dT_0}\,E_2(\eta_\nu-\tilde{\eta}_\nu)\,\frac{\partial T'}{\partial \eta}\,d\tilde{\eta} +\int_\eta^\infty \frac{dB_\nu(T_0)}{dT_0}\,E_2(\tilde{\eta}_\nu-\eta_\nu)\,\frac{\partial T'}{\partial \tilde{\eta}}\,d\tilde{\eta}+0(T'^2)\right] \qquad (20)$$

The quantity $dB_\nu(T_0)/dT_0$ is constant, since T_0 is constant. Therefore $dB_\nu(T_0)/dT_0$ could be taken outside the integrals. Instead, in order to consolidate the ν dependent factors in each term, α_{ν_0}, which is also constant, will be taken inside the integrals in the next rearrangement of equation (20).

The relation between η_ν and η is also needed only to lowest order in T'. Thus equation (15) can be written as

$$\eta_\nu=\int_0^\eta \frac{\alpha_{\nu_0}}{\alpha_0}\,d\hat{\eta}+0(T')$$

Since α_{ν_0}/α_0 is constant, it can be taken outside the integral and this becomes

$$\eta_\nu=\frac{\alpha_{\nu_0}}{\alpha_0}\int_0^\eta d\hat{\eta}+0(T')=\frac{\alpha_{\nu_0}}{\alpha_0}\,\eta+0(T') \qquad (21)$$

Substitution of equation (21) into (20) and rearrangement leads to the expression

$$Q_\nu=2\pi\alpha_{\nu_0}\,\frac{dB_\nu(T_0)}{dT_0}\,E_2\left(\frac{\alpha_{\nu_0}}{\alpha_0}\,\eta\right)(T'_w-T'|_{\eta=0}) -\int_0^\eta 2\pi\alpha_{\nu_0}\,\frac{dB_\nu(T_0)}{dT_0}\,E_2\left[\frac{\alpha_{\nu_0}}{\alpha_0}\,(\eta-\tilde{\eta})\right]\frac{\partial T'}{\partial \tilde{\eta}}\,d\tilde{\eta} +\int_\eta^\infty 2\pi\alpha_{\nu_0}\,\frac{dB_\nu(T_0)}{dT_0}\,E_0\left[\frac{\alpha_{\nu_0}}{\alpha_0}\,(\tilde{\eta}-\eta)\right]\frac{\partial T'}{\partial \tilde{\eta}}\,d\tilde{\eta}+0(T'^2) \qquad (22)$$

The function $2\pi\alpha_{\nu_0}\,\dfrac{dB_\nu(T_0)}{dT_0}\,E_2\left(\dfrac{\alpha_{\nu_0}}{\alpha_0}\,z\right)$ appearing in each term of equation (22) now contains all of the ν dependence. The subscript 0 indicates that this function is to be evaluated at the undisturbed condition of the gas. The integration over ν can therefore be completed for all time, without consideration of the temperature variations associated with particular problems. To this end, the function $F(\eta)$ is defined by the relation

$$CF(\eta)=\int_0^\infty 2\pi\alpha_{\nu_0}\,\frac{dB_\nu(T_0)}{dT_0}\,E_2\left(\frac{\alpha_{\nu_0}}{\alpha_0}\,\eta\right)d\nu \qquad (23)$$

The constant, C, can be chosen so as to obtain a formal similarity between equation (27) to be derived here and equation (33) of reference 1 (result for a grey gas). This is accomplished by imposing the requirement that, when α_{ν_0} is independent of ν (and hence equal to α_0), $F(\eta)$ will be equal to $E_2(\eta)$; that is,

$$CE_2(\eta)=\int_0^\infty 2\pi\alpha_0 \frac{dB_\nu(T_0)}{dT_0} E_2(\eta)d\nu$$

or

$$C=2\pi\alpha_0\int_0^\infty \frac{dB_\nu(T_0)}{dT_0} d_\nu=2\pi\alpha_0 \frac{d}{dT_0}\int_0^\infty B_\nu(T_0)d\nu$$

To evaluate the last expression, use can be made of the demonstration in a standard textbook that the Plank radiation law (eq. (8)) leads to Stefan's law; that is

$$\int_0^\infty B_\nu(T)d\nu=\frac{\sigma}{\pi} T^4 \tag{24}$$

where σ is the Stefan-Boltzmann constant (see, for example, ref. 28). By differentiation, it follows that

$$C=8\sigma T_0^3\alpha_0 \tag{25}$$

Use of this in equation (23) leads to

$$F(\eta;\, T_0)=\frac{\pi}{4\sigma T_0^3}\int_0^\infty \frac{\alpha_{\nu_0}}{\alpha_0}\frac{dB_\nu(T_0)}{dT_0} E_2\left(\frac{\alpha_{\nu_0}}{\alpha_0}\eta\right)d\nu \tag{26}$$

Here the parametric dependence of $F(\eta)$ on the temperature of the undisturbed gas, T_0, is indicated. Henceforth this will not be done.

When equation (22) is substituted into equation (6), and the order of integration interchanged, the result can be written in a form containing the combination $CF(z)$ in each term by the use of equation (23). Taking C outside the integrals, and replacing it by means of equation (25) finally leads to the result

$$Q=8\sigma T_0^3\alpha_0\bigg[F(\eta)(T'_w-T'|_{\eta=0}) \\ -\int_0^\infty F(\eta-\tilde{\eta})\frac{\partial\tilde{T}'}{\partial\tilde{\eta}}d\tilde{\eta}+\int_\eta^\infty F(\tilde{\eta}-\eta)\frac{\partial\tilde{T}'}{d\tilde{\eta}}d\tilde{\eta}\bigg] \tag{27}$$

where terms of order T'^2 are neglected. This result is formally identical to equation (33) of reference 1 when the latter is specialized to the case of a black wall. The only difference is that the $E_2(\eta)$ function, resulting from the grey-gas approximation in reference 1, is here replaced by the function $F(\eta)$ defined in equation (26). The definition of η, equation (13) is the same as that used in reference 1, where it appears as equation (22).

The foregoing procedure can be extended to higher order. It is then necessary to define additional functions, similar to $F(\eta)$, which involve integration over ν of integrands containing higher derivatives of $B_\nu(T_0)$ with respect to T_0, as well as derivatives of (α_ν/α) and (α_ν/ρ) with respect to T, also evaluated at $T=T_0$. The second-order equations are derived in appendix B.

Equation (27) is now in a form suitable for combination with equations (1) and (4) to obtain a single expression for the potential φ, but in addition a relation between η and x is needed. This can be found from the definition of η, given in equation (13). To lowest order, this becomes

$$\eta=\alpha_0[x-x_w(t)] \tag{28}$$

The earlier relations, to be used in the combination, are

$$\frac{\partial^2\varphi}{\partial t^2}-a_0^2\frac{\partial^2\varphi}{\partial x^2}=-\frac{(\gamma-1)}{\rho_0}Q \tag{1}$$

$$\frac{\partial T'}{\partial t}=-\frac{1}{R}\left(\frac{\partial^2\varphi}{\partial t^2}-\frac{a_0^2}{\gamma}\frac{\partial^2\varphi}{\partial x^2}\right) \tag{4}$$

Equation (28) can be used either to replace η in equation (27) or to replace x in equations (1) and (4), retaining η as an independent variable. If the results are extended to higher order, a difficulty associated with the transfer of boundary conditions is avoided by the use of η as independent variable. That approach is used in appendix B. However, such complications will be ignored here, and x used as a variable instead of η. Then $x_w(t)$ is taken to be a small quantity such that a displacement through this distance at any point in the field leads to higher order terms which are neglected in the first approximation. As discussed in reference 1, this is not a uniformly valid approximation because of the infinite derivative of the $E_2(z)$ function at a zero value of its argument. In the present impulsive piston problem there is a further difficulty arising from the fact that $x_w(t)$ will become large at large time. It can be shown that the result obtained by neglecting $x_w(t)$ is valid to lowest order (for values of x not too large) if x is measured from the wall rather than from a fixed origin. At large x, a straining of coordinates is required to render the results valid as in any acoustic theory. These points are further discussed in appendix B. Substitution of equation (28) into equation (27), with $x_w(t)$ equal to zero, yields

$$Q=8\sigma T_0^3\alpha_0\left\{F(\alpha_0 x)\,[T'_w(t)-T'(t,0)] -\int_0^x F[\alpha_0(x-\tilde{x})]\,\frac{\partial T'}{\partial \tilde{x}}\,d\tilde{x}+\int_x^\infty F[\alpha_0(\tilde{x}-x)]\,\frac{\partial T'}{\partial \tilde{x}}\,d\tilde{x}\right\} \tag{29}$$

Substitution of this into equation (1) and differentiation with respect to time leads to a relation containing $\frac{\partial^2 T'}{\partial x \partial t}$. Differentiation of equation (4) with respect to x relates the latter quantity to an expression in terms of φ so that T' can be elimated. The result is

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\varphi}{\partial t^2}-a_0^2\frac{\partial^2\varphi}{\partial x^2}\right)=\frac{8(\gamma-1)\sigma T_0^3\alpha_0}{R\rho_0}\left\{-F(\alpha_0 x)\left[R\frac{dT'_w}{dt}+\left(\frac{\partial^2\varphi}{\partial t^2}-\frac{a_0^2}{\gamma}\frac{\partial^2\varphi}{\partial x^2}\right)_{x=0}\right] -\int_0^x F[\alpha_0(x-\tilde{x})]\,\frac{\partial}{\partial\tilde{x}}\left(\frac{\partial^2\varphi}{\partial t^2}-\frac{a_0^2}{\gamma}\frac{\partial^2\varphi}{\partial\tilde{x}^2}\right)d\tilde{x} +\int_x^\infty F[\alpha_0(\tilde{x}-x)]\,\frac{\partial}{\partial\tilde{x}}\left(\frac{\partial^2\varphi}{\partial t^2}-\frac{a_0^2}{\gamma}\frac{\partial^2\varphi}{\partial\tilde{x}^2}\right)d\tilde{x}\right\} \tag{30}$$

This equation, derived for a gas with an arbitrary dependence of absorption coefficient on frequency, is formally identical to equation (38) of reference 1, when the latter is specialized to the case of a black wall. The only difference is that the E_2 function of reference 1 is here replaced by the function F, defined in equation (26). A result not restricted to the case of a black wall can be obtained by replacing the E_2 function with the F function in equation (38) of reference 1.

The boundary conditions associated with equation (30) are

$$\frac{\partial\varphi}{\partial x}(t,0)=u_w(t)=\text{given function of } t \tag{31}$$

$$T'_w(t)=\text{given function of } t \tag{32}$$

$$\varphi(t,\infty)=\text{finite quantity for all } t \tag{33}$$

II. APPROXIMATE SOLUTION FOR AN OSCILLATING PISTON

In reference 1 solutions of equation (30) corresponding to sinusoidal variations in wall velocity and temperature were obtained for the case of a grey gas ($F(z)=E_2(z)$) by approximating the E_2 function with an exponential. The same procedure can be followed in the general case, when $F(z)$ is not equal to $E_2(z)$, by setting

$$F(z)\approx me^{-nz} \tag{34}$$

The constants m and n can be chosen, as in reference 1, by making the approximation exact in the Rosseland limit of strong absorption and by further imposing a least squares fit. To obtain the Rosseland approximation correctly, it is necessary to match the first moment as follows:

$$\int_0^\infty F(z)zdz=\int_0^\infty me^{-nz}zdz=\frac{m}{n^2} \tag{35}$$

It can be shown that this is the correct criterion from equation (29). To do this, first integrate the integral terms by parts so that $\frac{\partial^2 T'}{\partial \bar{x}^2}$ appears under the integrals. A limiting process with α_0 going to infinity then shows the result to be proportional to a double integral of F. By a partial integration the latter quantity can be expressed as the first moment of F (see ref. 29).

Substitution of equation (9) into (26) and that into the last equation yields

$$\frac{m}{n^2}=\frac{\pi}{4\sigma T_0^3}\int_{z=0}^\infty\int_{\nu=0}^\infty \frac{\alpha_{\nu_0}}{\alpha_0}\frac{dB_\nu(T_0)}{dT_0}\int_{\mu=0}^1 \exp\left[-\frac{\alpha_{\nu_0}}{\alpha_0}\frac{z}{\mu}\right]d\mu d\nu z dz \tag{36}$$

With an interchange of the order of integration the z and μ integrations can be carried out to obtain

$$m=\frac{1}{3}n^2\int_0^\infty \frac{\alpha_{\nu_0}}{\alpha_0}\frac{dB_\nu(T_0)/dT_0}{\frac{4}{\pi}\sigma T_0^3}d\nu \tag{37}$$

When this is substituted into equation (34), and the value of n determined by a least squares fit of the result to equation (26), it is found that n must satisfy the relation

$$n\int_0^\infty \frac{dB_\nu/dT_0}{\frac{4}{\pi}\sigma T_0^3}\frac{\alpha_{\nu_0}}{\alpha_0}d\nu-4\int_0^\infty \frac{dB_\nu/dT_0}{\frac{4}{\pi}\sigma T_0^3\left(1+\frac{\alpha_0}{\alpha_{\nu_0}}n\right)}d\nu=0 \tag{38}$$

If α_{ν_0} is independent of ν, it can be seen that this reduces to

$$n-\frac{4}{1+n}=0$$

or $n=1.562$ as obtained in reference 1 for the case of a grey gas. In that case equation (37) reduces to $m=\frac{1}{3}n^2$, also obtained in reference 1. In the general case, when the dependence of α_{ν_0} on frequency is known, equation (38) can be solved numerically for n. The constant m can then be evaluated using equation (37). By redefining the quantities β and K appearing in equations (57 a, b) of reference 1 as

$$\beta=\frac{n\alpha_0 a_0}{\omega} \tag{39}$$

and

$$K=\frac{16(\gamma-1)\sigma T_0^3}{R\rho_0 a_0}\frac{m}{n} \tag{40}$$

one can show that the results of reference 1 in terms of β and K apply for a real gas when m and n are determined from equations (37) and (38). The validity of the exponential approximation is investigated in appendices C and D where it is shown that the resulting solutions lead to valid approximations of the physical quantities everywhere. In a small region near the wall, the higher derivatives of the physical quantities are singular. The exponential approximation does not reproduce this effect.

The results from reference 1 for the case of an oscillating wall are the following:

$$u(t,x)=\frac{a_0}{\gamma}\,\mathrm{Re}\left\{\left[c_1C_1\exp\left(\frac{c_1\omega x}{a_0}\right)+c_2C_2\exp\left(\frac{c_2\omega x}{a_0}\right)\right]e^{i\omega t}\right\} \quad (41)$$

$$\mathrm{Re}\,[(c_1C_1+c_2C_2)e^{i\omega t}]=\frac{\gamma u_w(t)}{a_0} \quad (42)$$

$$\mathrm{Re}\left\{-i\left[\left(1+\frac{c_1^2}{\gamma}\right)\left(\frac{\beta}{\beta+c_1}\right)C_1+\left(1+\frac{c_2^2}{\gamma}\right)\left(\frac{\beta}{\beta+c_2}\right)C_2\right]e^{i\omega t}\right\}=\frac{T'_w(t)}{T_0} \quad (43)$$

where

$$a_0=\gamma RT_0$$

$$\left.\begin{matrix}c_1\\c_2\end{matrix}\right\}=-\left[\frac{-(1-\beta^2-iK\beta)\mp\sqrt{(1-\beta^2-iK\beta)^2+4\beta^2(1-iK\beta/\gamma)}}{2(1-iK\beta/\gamma)}\right]^{\frac{1}{2}} \quad (44)$$

Equation (44) is the solution of a fourth degree characteristic equation which results from substituting a complex exponential form into the integro-differential equation. The derivation of these relations is contained in reference 1 and is further discussed in appendix C herein.

Equations (41), (42), and (43) apply when the wall velocity $u_w(t)$ and the wall perturbation temperature $T'_w(t)$ are sinusoidal functions of time with radian frequency ω and arbitrary phase. When this is the case, equations (42) and (43) can be solved for the complex amplitudes C_1 and C_2. Equation (41) then represents two damped sinusoidal traveling waves. The damping constants and wave speeds are determined by the complex constants c_1 and c_2. These are given in terms of the radian frequency ω, the basic physical parameters, and the approximation constants m and n by equations (39), (40), and (44).

The variations of the damping constants and wave speeds through the whole range of parameters are discussed in reference 1. As an example of these results, the wave speeds v_1 and v_2 are plotted in figure 2 as functions of the parameter $\omega/n\alpha_0a_0$ for $\gamma=7/5$ and $K=4$. In the figure, the dimensionless quantities v_1/a_0 and $0.01\,v_2/a_0$ are plotted to avoid specifying a_0 which is the isentropic speed of sound in the undisturbed gas. It is seen that v_1 is equal to a_0 at low frequency and also at high frequency. At intermediate frequencies the wave speed approaches the isothermal signal velocity $a_0/\sqrt{\gamma}$. Since the speed of this wave does not deviate greatly from the isentropic speed of sound, it has been referred to as a modified-classical wave. The other wave speed, v_2, varies between zero and the velocity of light (taken to be infinite), depending on the frequency. The term of equation (41) corresponding to v_2 has been referred to in reference 1 as a radiation-induced wave. The wave speeds are determined from the values of the complex constants c_1 and c_2 by the relations

$$v_1/a_0=-[\text{Imaginary Part }(c_1)]^{-1} \quad (45)$$

$$v_2/a_0=-[\text{Imaginary Part }(c_2)]^{-1} \quad (46)$$

Figure 2 was plotted from data obtained by electronic machine evaluation of equations (44)–(46).

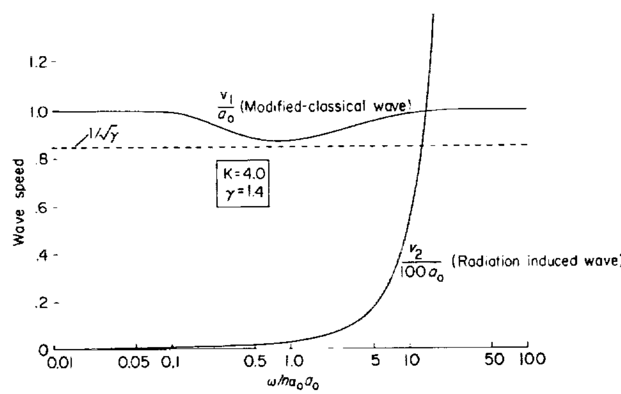


FIGURE 2.—Wave speeds versus frequency of oscillation.

III. IMPULSIVE PISTON SOLUTION

The response of a radiating gas to an impulse motion of a wall is governed by the integro-partial differential equation (30). With the exponential approximation (eq. (34)) of the attenuation factor $F(z)$, this is

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\varphi}{\partial t^2}-a_0^2\,\frac{\partial^2\varphi}{\partial x^2}\right)=\frac{8(\gamma-1)\sigma T_0^3}{R\rho_0}\,\alpha_0 m$$
$$\left\{-e^{-n\alpha_0 x}\left[R\,\frac{dT'_w}{dt}+\left(\frac{\partial^2\varphi}{\partial t^2}-\frac{a_0^2}{\gamma}\,\frac{\partial^2\varphi}{\partial x^2}\right)_{x=0}\right]\right.$$
$$-\int_0^x e^{-n\alpha_0(x-\tilde{x})}\frac{\partial}{\partial\tilde{x}}\left(\frac{\partial^2\varphi}{\partial t^2}-\frac{a_0^2}{\gamma}\,\frac{\partial^2\varphi}{\partial x^2}\right)d\tilde{x}$$
$$\left.+\int_x^\infty e^{-n\alpha_0(\tilde{x}-x)}\frac{\partial}{\partial\tilde{x}}\left(\frac{\partial^2\varphi}{\partial t^2}-\frac{a_0^2}{\gamma}\,\frac{\partial^2\varphi}{\partial\tilde{x}^2}\right)d\tilde{x}\right\}\qquad(47)$$

where φ is a potential from which the gas perturbation velocity, pressure, temperature, and density can be found using equations (2)–(5). The boundary conditions to be satisfied are

$$\frac{\partial\varphi}{\partial x}\,(t,0)=u_w(t)=\begin{cases}0 & (t<0)\\ U & (t>0)\end{cases}\qquad(48)$$

$$T'_w(t)=\begin{cases}0 & (t<0)\\ \Theta & (t>0)\end{cases}\qquad(49)$$

The initial condition

$$\varphi(t,x)=0\qquad(t<0)\qquad(50)$$

is also imposed.

In the results to be given, the perturbation wall temperature will be taken zero for all time ($\Theta=0$) and the wall velocity at $t>0$ will be considered constant. However, the required relations for more general cases will be developed up to a point.

Solutions corresponding to general time-dependent boundary conditions can be obtained by superposition of the oscillating piston results set down in the previous section. As they stand, equations (41), (42) and (43) apply only when $u_w(t)$ and $T'_w(t)$ are sinusoidal functions of time with radian frequency ω. For the general time-dependent case, the right side of equation (41) and the left sides of equations (42) and (43) should be integrated over all values of ω from zero to ∞ as follows:

$$u(t,x)=\frac{a_0}{\gamma}\,\mathrm{Re}\int_0^\infty\left(c_1C_1e^{c_1\omega x/a_0}+c_2C_2e^{c_2\omega x/a_0}\right)e^{i\omega t}d\omega\qquad(51)$$

$$\mathrm{Re}\int_0^\infty\left(c_1C_1+c_2C_2\right)e^{i\omega t}d\omega=\frac{\gamma}{a_0}\,u_w(t)\qquad(52)$$

$$\mathrm{Re}\left\{-i\int_0^\infty\left[\left(1+\frac{c_1^2}{\gamma}\right)\left(\frac{\beta}{\beta+c_1}\right)C_1+\left(1+\frac{c_2^2}{\gamma}\right)\left(\frac{\beta}{\beta+c_2}\right)C_2\right]e^{i\omega t}d\omega\right\}=\frac{T'_w(t)}{T_0}\qquad(53)$$

It is expedient to make use of Fourier-transform theory in evaluating C_1 and C_2 from equations (52) and (53). To this end, the Fourier transform for an arbitrary function $f(t,x)$ is defined by the relation

$$\bar{f}(\omega,x)=\frac{1}{\pi}\int_{-\infty}^\infty f(t,x)e^{-i\omega t}dt\qquad(54)$$

where the bar over the function indicates that it is a transform. The inverse relation

$$f(t,x)=\frac{1}{2}\int_{-\infty}^\infty\bar{f}(\omega,x)e^{i\omega t}d\omega\qquad(55)$$

follows (see, for example, ref. 30). This definition differs from the conventional one in that the $(-i)$ is usually associated with the inversion integral rather than the transform as it is here. The reason for the present choice is to avoid the necessity of changing the sign in the appropriate places in all of the results of reference 1. When $f(t,x)$ is real, it can be shown that the real part of $\bar{f}(\omega,x)$ must be an even function of ω and the imaginary part odd. When $f(t)$ is also zero for $t<0$, equations (54) and (55) can be written alternatively as

$$\bar{f}(\omega,x)=\frac{1}{\pi}\int_0^\infty f(t,x)e^{-i\omega t}dt\qquad(56)$$

$$f(t,x)=\mathrm{Re}\int_0^\infty\bar{f}(\omega,x)e^{i\omega t}d\omega\qquad(57)$$

By choosing $f(t,x)$ in these forms to be $\frac{\gamma}{a_0}\,u_w(t)$ and comparing equation (52) with (57), it is seen that equation (52) corresponds to the relation

$$c_1C_1+c_2C_2=\frac{\gamma}{a_0}\,\overline{u}_w(\omega) \tag{58}$$

Similarly equation (53) becomes

$$-i\left[\left(1+\frac{c_1^2}{\gamma}\right)\left(\frac{\beta}{\beta+c_1}\right)C_1+\left(1+\frac{c_2^2}{\gamma}\right)\left(\frac{\beta}{\beta+c_2}\right)C_2\right]=\frac{\overline{T}'_w(\omega)}{T_0} \tag{59}$$

When $u_w(t)$ and $T'_w(t)$ are given functions of t that are zero for $t<0$, their transforms can be found according to equation (56) and are given by the relations

$$\overline{u}_w(\omega)=\frac{1}{\pi}\int_0^\infty u_w(t)e^{-i\omega t}dt \tag{60}$$

$$\overline{T}'_w(\omega)=\frac{1}{\pi}\int_0^\infty T'_w(t)e^{-i\omega t}dt \tag{61}$$

These results, substituted into equations (58) and (59), lead to two simultaneous equations which can be solved for the quantities C_1 and C_2 needed in the integral expressions for velocity, temperature, and pressure, to be given in equations (62)–(64). The quantities c_1, c_2, and β appearing in equations (58) and (59) are the same functions of ω as in the oscillating piston solutions and are given by equations (39) and (44).

The integral expression for the velocity, equation (51), can also be written as

$$u(t,x)=\frac{a_0}{2\gamma}\int_{-\infty}^\infty (c_1C_1e^{c_1\omega x/a_0}+c_2C_2e^{c_2\omega x/a_0})e^{i\omega t}d\omega$$
$$=\frac{a_0}{\gamma}\,\mathrm{Re}\int_0^\infty (c_1C_1e^{c_1\omega x/a_0}+c_2C_2e^{c_2\omega x/a_0})e^{i\omega t}d\omega \tag{62}$$

The second equality follows from the symmetry of the integrand.

Similar expressions for the pressure and temperature can be found using equations (2), (3), (4) and (62). The results are

$$p'(t,x)=-\frac{p_0}{2}\,i\int_{-\infty}^\infty (C_1e^{c_1\omega x/a_0}+C_2e^{c_2\omega x/a_0})e^{i\omega t}d\omega$$
$$=-p_0\,\mathrm{Re}\int_0^\infty (C_1e^{c_1\omega x/a_0}+C_2e^{c_2\omega x/a_0})e^{i\omega t}d\omega \tag{63}$$

and

$$T'(t,x)=-\frac{T_0}{2}\,i\int_{-\infty}^\infty \left[\left(1+\frac{c_1^2}{\gamma}\right)C_1e^{c_1\omega x/a_0}+\left(1+\frac{c_2^2}{\gamma}\right)C_2e^{c_2\omega x/a_0}\right]e^{i\omega t}d\omega$$
$$=-T_0\,\mathrm{Re}\; i\int_0^\infty \left[\left(1+\frac{c_1^2}{\gamma}\right)C_1e^{c_1\omega x/a_0}+\left(1+\frac{c_2^2}{\gamma}\right)C_2e^{c_2\omega x/a_0}\right]e^{i\omega t}d\omega \tag{64}$$

In the derivation of these relations, differentiations and integrations with respect to t and x are taken inside the integral with respect to ω. Also use is made of the fact that the perturbation quantities are zero at $t<0$. These steps can be justified only under certain conditions which require discussion. It is necessary that the integrals with respect to ω be integrable at least in the sense of the Cauchy principal value. For example, if the functions $u_w(t)$ and $T'_w(t)$ are such that the transforms $\overline{u}'_w(\omega)$ or $\overline{T}'_w(\omega)$ are singular at any point on the path of integration (the real axis in the complex ω plane), then these singularities must be circumnavigated along infinitesimal semicircles either above or below the real axis in the complex ω plane. In the present problem, it can be shown that the path of integration in equations (62), (63) and (64) should pass below any singularities on the real axis to insure that the perturbation quantities will be zero for $t<0$. The demonstration is somewhat complicated by the existence of branch points. Equation (44) can be written as

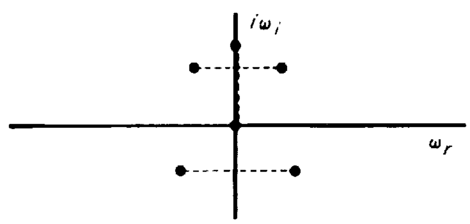
$$\left.\begin{matrix}c_1\\c_2\end{matrix}\right\}=-\left[\frac{-(\omega^2-n^2\alpha_0^2a_0^2-iKn\alpha_0a_0\omega)\mp\sqrt{(\omega^2-n^2\alpha_0a_0^2-iKn\alpha_0a_0\omega)^2+4n^2\alpha_0^2a_0^2\omega(\omega-iKn\alpha_0a_0/\gamma)}}{2\omega(\omega-iKn\alpha_0a_0/\gamma)}\right]^{\frac{1}{2}} \tag{65}$$

The quantities C_1 and C_2 appearing in the integrands can be found by solution of equations (58) and (59). Upon replacing the quantity β with the aid of equation (39), the results are

$$C_1=\frac{-\left(1+\frac{c_2^2}{\gamma}\right)\left(\frac{n\alpha_0 a_0}{n\alpha_0 a_0+\omega c_2}\right)\frac{\gamma}{a_0}\,\overline{u}_w(\omega)+ic_2\frac{\overline{T}'_w(\omega)}{T_0}}{\left(1+\frac{c_1^2}{\gamma}\right)\left(\frac{n\alpha_0 a_0}{n\alpha_0 a_0+\omega c_1}\right)c_2-\left(1+\frac{c_2^2}{\gamma}\right)\left(\frac{n\alpha_0 a_0}{n\alpha_0 a_0+\omega c_2}\right)c_1} \tag{66}$$

$$C_2=\frac{\left(1+\frac{c_1^2}{\gamma}\right)\left(\frac{n\alpha_0 a_0}{n\alpha_0 a_0+\omega c_1}\right)\frac{\gamma}{a_0}\,\overline{u}_w(\omega)-ic_1\frac{\overline{T}'_w(\omega)}{T_0}}{\left(1+\frac{c_1^2}{\gamma}\right)\left(\frac{n\alpha_0 a_0}{n\alpha_0 a_0+\omega c_1}\right)c_2-\left(1+\frac{c_2^2}{\gamma}\right)\left(\frac{n\alpha_0 a_0}{n\alpha_0 a_0+\omega c_2}\right)c_1} \tag{67}$$

It can be seen from equation (65) that there are six branch points in the expression for c_1 and c_2. Three of these are in the upper half of the complex ω plane and one at the origin (see sketch).



The remaining two are in the lower half-plane. The quantities c_1 and c_2 will be single-valued functions of ω if a branch cut is introduced connecting the four branch points on or above the real axis, and a separate branch cut is introduced connecting the two branch points in the lower half plane. In the following, the properties of c_1 and c_2 in the lower half-plane only are of interest. It can be seen that c_1 on one side of the branch cut in the lower half-plane becomes c_2 on the other side and vice versa. From equations (66) and (67) it follows that the same is true of C_1 and C_2. Since the integrands of equations (62), (63), and (64) are unchanged by an interchange of the subscripts 1 and 2, the branch cut in the lower half plane can be dispensed with. In general, there may be additional regions of nonanalyticity arising from $\overline{u}_w(\omega)$ and $\overline{T}'_w(\omega)$. For the specific problem to be considered, there are not. At this point we will specialize to those cases where $\overline{u}_w(\omega)$ and $\overline{T}'_w(\omega)$ are analytic in the lower half-plane. Then, altogether, one can conclude that the integrands of equations (62)–(64) are analytic in the entire lower half-plane, but contain singularities and branch points on and above the real axis.

The Cauchy integral theorem can now be used to show that the path of integration in equations (62)–(64) should be taken below any singularities on the real axis to insure that the perturbation quantities will be zero for $t<0$. Let us assume that the properties of $\overline{u}_w(\omega)$ and $\overline{T}'_w(\omega)$ are such that the parts of the integrands excluding $e^{i\omega t}$ in equations (62)–(64) are zero at infinity. Then for t less than zero, each line integral involved can be closed with a semicircle at infinity in the lower half-plane without adding anything to the value of the integral. The resulting closed contour will not enclose any poles so long as the path of integration along the real axis is taken to be below any singularities on the real axis. In that case the value of each of the original line integrals is zero for $t<0$.

We will now specialize to the case of impulsive motion of the wall with wall temperature held constant. The boundary conditions for this case can be written as

$$u_w(t)=\begin{cases}0 & (t<0)\\ \lim\limits_{\epsilon\to 0} Ue^{-\epsilon t} & (t>0)\end{cases} \tag{68}$$

$$T'_w(t)=0 \qquad (\text{all } t) \tag{69}$$

Substitution of these relations into equation (56) yields

$$\overline{u}_w(\omega)=\frac{1}{\pi}U\lim_{\epsilon\to 0}\int_0^\infty e^{-\epsilon t}e^{-i\omega t}dt \tag{70}$$

and

$$\overline{T}(\omega)=0 \tag{71}$$

Evaluation of equation (70) leads to the result

$$\overline{u}_w(\omega)=\frac{1}{\pi}U\lim_{\epsilon\to 0}\left(\frac{1}{\epsilon+i\omega}\right)=-i\frac{U}{\pi\omega} \tag{72}$$

At this point, it is helpful to note that there is a basic similarity in the problem. By the introduction of a set of dimensionless variables, it is possible to present the results in a form which is independent of the values of the radiation absorption coefficient, α_0, the approximation parameters

m, n, and the gas properties a_0, R. An appropriate set of similarity variables is the following:

$$\tau=\sqrt{\frac{2}{\gamma+1}}n\alpha_0 a_0 t \tag{73}$$

$$\xi=\sqrt{\frac{2}{\gamma+1}}n\alpha_0 x \tag{74}$$

$$\nu=\omega/\sqrt{\frac{2}{\gamma+1}}n\alpha_0 a_0 \tag{75}$$

$$k=\sqrt{\frac{\gamma+1}{2}}K/\gamma=16\sqrt{\frac{\gamma+1}{2}}(\gamma-1)\frac{m}{n}\frac{\sigma T_0^3}{\gamma R\rho_0 a_0} \tag{76}$$

$$A_j=i\pi\nu\sqrt{\frac{2}{\gamma+1}}n\alpha_0 a_0\left(\frac{a_0}{\gamma U}\right)c_j C_j \qquad (j=1,2) \tag{77}$$

Using these equations, the expressions for the velocity, pressure, and temperature become

$$\frac{u(\tau,\xi)}{U}=-\frac{i}{2\pi}\int_{-\infty}^{\infty}(A_1e^{c_1\nu\xi}+A_2e^{c_2\nu\xi})e^{i\nu\tau}\frac{d\nu}{\nu} \tag{78}$$

$$\frac{p'(\tau,\xi)}{\rho_0 a_0 U}=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{A_1}{c_1}e^{c_1\nu\xi}+\frac{A_2}{c_2}e^{c_2\nu\xi}\right)e^{i\nu\tau}\frac{d\nu}{\nu} \tag{79}$$

$$\frac{RT'(\tau,\xi)}{a_0U}=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\left(1+\frac{c_1^2}{\gamma}\right)\frac{A_1}{c_1}e^{c_1\nu\xi}+\left(1+\frac{c_2^2}{\gamma}\right)\frac{A_2}{c_2}e^{c_2\nu\xi}\right]e^{i\nu\tau}\frac{d\nu}{\nu} \tag{80}$$

The wave speed parameters c_1, c_2, and the wave amplitude parameters A_1, A_2 are given by the relations

$$\left.\begin{matrix}c_1\\c_2\end{matrix}\right\}=-\left[\frac{-\left(\nu^2-i\gamma k\nu-\frac{\gamma+1}{2}\right)\pm\sqrt{\left(\nu^2-i\gamma k\nu-\frac{\gamma+1}{2}\right)^2+4\left(\frac{\gamma+1}{2}\right)\nu(\nu-ik)}}{2\nu(\nu-ik)}\right]^{1/2} \tag{81}$$

$$A_1=1\Bigg/\left[1-\left(\frac{1+\frac{c_1^2}{\gamma}}{1+\frac{c_2^2}{\gamma}}\right)\left(\frac{1+\sqrt{\frac{2}{\gamma+1}}\nu c_2}{1+\sqrt{\frac{2}{\gamma+1}}\nu c_1}\right)\frac{c_2}{c_1}\right] \tag{82}$$

$$A_2=-\left(\frac{1+\frac{c_1^2}{\gamma}}{1+\frac{c_2^2}{\gamma}}\right)\left(\frac{1+\sqrt{\frac{2}{\gamma+1}}\nu c_2}{1+\sqrt{\frac{2}{\gamma+1}}\nu c_1}\right)\frac{c_2}{c_1}\Bigg/\left[1-\left(\frac{1+\frac{c_1^2}{\gamma}}{1+\frac{c_2^2}{\gamma}}\right)\left(\frac{1+\sqrt{\frac{2}{\gamma+1}}\nu c_2}{1+\sqrt{\frac{2}{\gamma+1}}\nu c_1}\right)\right] \tag{83}$$

As discussed earlier, the line integrals in equations (78)–(80) should be distorted from the real axis near $\nu=0$ so as to pass below the singularity at the origin.

The task before us now is to evaluate the integrals in equations (78)–(80). This can be done in closed form only for certain limiting cases or by approximation. The methods used will be demonstrated for two cases in this section, and the remainder will be treated in appendices E and F. One question which arises is whether discontinuities occur in the flow field as a result of the discontinuous wall velocity. Such a discontinuity might occur at $\tau=0$. It will first be shown that this is not the case. For this purpose we shall concentrate on the expression for the velocity given in equation (78).

It was demonstrated earlier by means of contour integration that equation (78) will lead to $u/U=0$ for $\tau<0$. Therefore, the quantity $e^{i\nu\tau}$ can be replaced by $e^{i\nu\tau}-e^{-i\nu|\tau|}$ since the added term will be zero except possibly at $\tau=0$. Then equation (78) can be replaced by

$$\frac{u(\tau,\xi)}{U}=-\frac{i}{2\pi}\int_{-\infty}^{\infty}(A_1e^{c_1\nu\xi}+A_2e^{c_2\nu\xi})(e^{i\nu\tau}-e^{-i\nu|\tau|})\frac{d\nu}{\nu}$$

$$=\begin{cases}0 & (\tau<0)\\ \frac{1}{\pi}\int_{-\infty}^{\infty}(A_1e^{c_1\nu\xi}+A_2e^{c_2\nu\xi})\frac{\sin\nu\tau}{\nu}d\nu & (\tau>0)\end{cases} \tag{84}$$

It can be seen that the real part of the integrand in this expression is an even function of ν, and the imaginary part is an odd function so that equation (84) can be written as

$$\frac{u(\tau,\xi)}{U}=\frac{2}{\pi}\int_0^\infty \mathrm{Re}\,(A_1e^{c_1\nu\xi}+A_2e^{c_2\nu\xi})\,\frac{\sin\nu\tau}{\nu}\,d\nu \qquad (\tau>0) \quad (85)$$

For slightly positive values of τ, this can be written

$$\frac{u(0+,\xi)}{U}=\frac{2}{\pi}\lim_{\tau\to 0+}\int_0^{1/\sqrt{\tau}}\mathrm{Re}\,(A_1e^{c_1\nu\xi}+A_2e^{c_2\nu\xi})\,\frac{\sin\nu\tau}{\nu}\,d\nu + \frac{2}{\pi}\lim_{\tau\to 0+}\int_{1/\sqrt{\tau}}^\infty \mathrm{Re}\,(A_1e^{c_1\nu\xi}+A_2e^{c_2\nu\xi})\,\frac{\sin\nu\tau}{\nu}\,d\nu \quad (86)$$

The integral is broken into two parts here to promote simplifications in each part. The point of division is, to some extent, arbitrary. By expansion of $\sin\nu\tau$ in a power series and use of the fact that the quantity Re $[A_1 \exp(c_1\nu\xi)+A_2\exp(c\nu\xi)]$ is bounded, it can be seen that the first term in this expression for $u(0+,\xi)/U$ is, at most, proportional to $\sqrt{\tau}$ multiplied by a convergent series and hence vanishes in the limit as τ goes to zero. Then equation (86) becomes

$$\frac{u(0+,\xi)}{U}=\frac{2}{\pi}\lim_{\tau\to 0+}\int_{1/\sqrt{\tau}}^\infty \mathrm{Re}\,(A_1e^{c_1\nu\xi}+A_2e^{c_2\nu\xi})\,\frac{\sin\nu\tau}{\nu}\,d\nu \quad (87)$$

If the quantity $\mathrm{Re}(A_1e^{c_1\nu\xi}+A_2e^{c_2\nu\xi})$ is expanded for large ν, it can be seen that only the lowest order terms can contribute to the integral in the limit.

For later use, the wave speed parameters c_1 and c_2, given in equation (81), can be partially expanded for both large and small ν in the forms

$$c_1=-i\left[1-i\,\frac{(\gamma-1)k\nu}{\nu^2-i\gamma k\nu+\frac{\gamma+1}{2}}-\frac{\frac{\gamma+1}{2}(\gamma-1)^2k^2\nu^2}{\left(\nu^2-i\gamma k\nu+\frac{\gamma+1}{2}\right)^3}+\begin{cases}0(\nu^3)\,(\text{small }\nu)\\0(\nu^{-7})\,(\text{large }\nu)\end{cases}\right]^{1/2} \quad (88)$$

$$c_2=-\sqrt{\frac{(\gamma+1)/2}{\nu(\nu-ik)}}\left[1+\frac{i(\gamma-1)k\nu}{\nu^2-i\gamma k\nu+\frac{\gamma+1}{2}}-\frac{\left(\frac{\gamma+1}{2}\right)(\gamma-1)^2k^2\nu^2}{\left(\nu^2-i\gamma k\nu+\frac{\gamma+1}{2}\right)^3}+\begin{cases}0(\nu^2)\,(\text{small }\nu)\\0(\nu^{-9})\,(\text{large }\nu)\end{cases}\right]^{1/2} \quad (89)$$

This result is obtained by rearrangement of the quantity under the inner radical in the form

$$\left(\nu^2-i\gamma k\nu-\frac{\gamma+1}{2}\right)^2+4\left(\frac{\gamma+1}{2}\right)\nu(\nu-ik) =\left(\nu^2-i\gamma k\nu+\frac{\gamma+1}{2}\right)^2+4\left(\frac{\gamma+1}{2}\right)(\gamma-1)ik\nu$$

and by expanding about a zero value of the second term on the right. Further expansion of equations (88) and (89) for large ν leads to the relations

$$c_1=-i-\frac{\gamma-1}{2}\frac{k}{\nu}+0\left(\frac{1}{\nu^2}\right) \qquad k \text{ finite} \quad (90)$$

$$c_2=-\sqrt{\frac{\gamma+1}{2}}\frac{1}{\nu}+0\left(\frac{1}{\nu^2}\right) \qquad k \text{ finite} \quad (91)$$

Substitution of these expressions into equations (82) and (83), and expansion of the result for large ν yields

$$A_1=1+0\left(\frac{1}{\nu^3}\right) \qquad k \text{ finite} \quad (92)$$

$$A_2=0\left(\frac{1}{\nu^3}\right) \qquad k \text{ finite} \quad (93)$$

Use of equations (90)–(93) in equation (87) leads to

$$\frac{u(0+,\xi)}{U}=\lim_{\tau\to 0+}\frac{2}{\pi}\int_{1/\sqrt{\tau}}^\infty \mathrm{Re}\,e^{-i\nu\xi}e^{-\left(\frac{\gamma-1}{2}\right)k\xi}\left[1+0\left(\frac{1}{\nu}\right)\right]\frac{\sin\nu\tau}{\nu}\,d\nu$$

It can be seen that the contribution of the term containing $0\left(\frac{1}{\nu}\right)$ is zero in the limit as τ goes to zero and the integral can be written as

$$\frac{u(0+,\xi)}{U}=\exp\left[-\left(\frac{\gamma-1}{2}\right)k\xi\right]\lim_{\tau\to 0+}\frac{2}{\pi}\int_{1/\sqrt{\tau}}^\infty \frac{\cos(\nu\xi)\sin(\nu\tau)}{\nu}\,d\nu$$

By the same procedure as before it follows that the contribution to the integral from this integrand in an interval 0 to $1/\sqrt{\tau}$ is zero in the limit. Adding this interval, again using the symmetry, and expressing the cosine and sine in exponential form, the integral can be written as

$$\frac{u(0+,\xi)}{U}=e^{-\left(\frac{\gamma-1}{2}\right)k\xi}\lim_{\tau\to 0+}\left(\frac{-i}{4\pi}\right)\int_{-\infty}^\infty [e^{i\nu(\tau+\xi)}+e^{i\nu(\tau-\xi)}-e^{-i\nu(\tau-\xi)}-e^{-i\nu(\tau+\xi)}]\,\frac{d\nu}{\nu}$$

If we take the line integral to pass below the origin, each exponential term can be evaluated separately. This is done by closing the contour with a semicircle at infinity in the lower or upper half plane, depending on the sign of the argument of the exponential, and using the calculus of residues. The result is the expression

$$\frac{u(0+,\xi)}{U}=e^{-\left(\frac{\gamma-1}{2}\right)k\xi}\lim_{\tau\to 0+}\frac{1}{2}\left(\frac{\tau+\xi}{|\tau+\xi|}+\frac{\tau-\xi}{|\tau-\xi|}\right)$$

To evaluate this, the value of ξ must be specified relative to τ before taking the limit. When this is done, it can be seen that

$$\frac{u(0+,\xi)}{U}=\begin{cases}0 & 0<\tau<\xi\\ 1 & 0<\xi<\tau\end{cases} \tag{94}$$

Since the expansion for large ν used in the derivation is valid only for finite k, this result may not apply in the limit as $k\to\infty$, although it does apply at all other values of k. In fact, the same result can be obtained for infinite k, but the proof requires a more detailed procedure which will not be given here.

We have found that the gas velocity is zero at slightly positive values of τ except at the wall ($\xi=0$), where the gas velocity is equal to the wall velocity. Corresponding results can be found for the perturbation pressure and temperature. These results are

$$\frac{p'(0+,\xi)}{\rho_0 a_0 U}=\begin{cases}0 & 0<\tau<\xi\\ 1 & 0<\xi<\tau\end{cases} \tag{95}$$

$$\frac{RT'(0+,\xi)}{a_0 U}=\begin{cases}0 & 0<\tau<\xi\\ (\gamma-1)/\gamma & 0<\xi<\tau\end{cases} \tag{96}$$

By a similar procedure, it can be shown that jumps in velocity, pressure, and temperature occur at $\tau=\xi$ for all finite ξ. For the velocity, the quantity to be evaluated is

$$\lim_{\epsilon\to 0}\left[\frac{u(\xi+\epsilon,\xi)}{U}-\frac{u(\xi-\epsilon,\xi)}{U}\right]=\lim_{\epsilon\to 0}\left(\frac{-i}{2\pi}\right)\int_{-\infty}^{\infty}(e^{i\nu\epsilon}-e^{-i\nu\epsilon})\,[A_1e^{(c_1+i)\nu\xi}+A_2e^{(c_2+i)\nu\xi}]\frac{d\nu}{\nu}$$

The results are

$$\lim_{\epsilon\to 0}\left[\frac{u(\xi+\epsilon,\xi)}{U}-\frac{u(\xi-\epsilon,\xi)}{U}\right]=\exp\left[-\left(\frac{\gamma-1}{2}\right)k\xi\right] \tag{97}$$

$$\lim_{\epsilon\to 0}\left[\frac{p'(\xi+\epsilon,\xi)}{\rho_0 a_0 U}-\frac{p'(\xi-\epsilon,\xi)}{\rho_0 a_0 U}\right]=\exp\left[-\left(\frac{\gamma-1}{2}\right)k\xi\right] \tag{98}$$

$$\lim_{\epsilon\to 0}\left[\frac{RT'(\xi+\epsilon,\xi)}{a_0 U}-\frac{RT'(\xi-\epsilon,\xi)}{a_0 U}\right]=\left(\frac{\gamma-1}{\gamma}\right)\exp\left[-\left(\frac{\gamma-1}{2}\right)k\xi\right] \tag{99}$$

Equations (97)–(99) indicate that the jumps in velocity, pressure, and temperature at $\tau=\xi$ decay exponentially with distance from the wall. This finding is qualitatively similar to that for a gas in chemical or vibrational nonequilibrium in the absence of radiation (ref. 31). However, in the present case where the effect is due to radiative heat transfer, the disturbance is not zero ahead of the jump ($\tau<\xi$) as it is for chemical nonequilibrium. This will be demonstrated later.

In reference 10, an expression is derived for the velocity field far from the wall in the case of chemical nonequilibrium.[1] The same can be done in the present problem by expanding the integrands in equations (78)–(80) for small values of ν and demonstrating that the contribution to the integral from other values of ν is zero. The evaluation for this and for other limiting cases is given in appendix E.

To obtain a qualitative view of the over-all flow field, either approximation or machine computation must be resorted to. Both methods have been used to evaluate the velocity disturbance for the present problem. An approximate closed-form solution is derived in appendix F.

[1] In references 10 and 31, the problem treated is that of the steady supersonic flow past a wedge. For chemical or vibrational nonequilibrium there is a direct analogy with the corresponding one-dimensional unsteady flow problem. Strictly speaking, such an analogy does not exist in the case of radiative heat transfer because of the directional properties of the radiation intensity.

The result is

$$\frac{u_A}{U}(\tau,\xi)=0 \qquad (\tau<0)$$

$$\frac{u_A}{U}(\tau,\xi)=\frac{1}{2}\left(1-e^{-X/b^2}\right)\left[\operatorname{erf}\left(\frac{\tau-\xi}{2\sqrt{X}}\right)+\operatorname{erf}\left(\frac{\tau+\xi}{2\sqrt{X}}\right)\right]$$
$$+\frac{1}{2}\left(e^{-X/b^2}-e^{-X}\right)\left\{\operatorname{erf}\left[\frac{b(\tau-\xi)-X}{2\sqrt{X}}\right]+\operatorname{erf}\left[\frac{b(\tau+\xi)+X}{2\sqrt{X}}\right]\right\}$$
$$+\frac{1}{2}e^{-X}\left[1+\frac{(\tau-\xi)}{|\tau-\xi|}\right] \qquad (\tau>0) \quad (100)$$

where

$$X=(\sqrt{\gamma}-1)k\xi \quad (101)$$

$$b=k/2+\sqrt{(k/2)^2+1} \quad (102)$$

The over-all velocity field will be described in the next section.

IV. RESULTS AND DISCUSSION

Equation (100) is a closed-form uniformly valid approximation for the linearized velocity response to an impulsive motion of the wall. A comparison of this result with numerical evaluations is made in appendix G. Equation (100) provides a good qualitative summary of the results of the numerical investigation and will now be used for that purpose. An evaluation of this equation for an intermediate value of the radiation parameter ($k=3.0$) is shown in figure 3. The value $k=3.0$ corresponds roughly to the value $K=4.0$ used in figure 2. The ratio of specific heats γ is taken equal to 7/5, but the results would be qualitatively similar for any γ between 1 and 5/3.

Figure 3 is a plot of gas velocity divided by piston velocity as a function of time τ and distance from the wall ξ. These results apply at all values of the mean radiation absorption coefficient, α_0, by virtue of the basic similarity and use of the radiation mean free path (α_0^{-1}) as the unit of length. The dimensionless time, τ, and distance, ξ, are defined in equations (73) and (74) which are $\tau=\sqrt{2/(\gamma+1)}n\alpha_0 a_0 t$ and $\xi=\sqrt{2/(\gamma+1)}n\alpha_0 x$. Since $\sqrt{2/(\gamma+1)}n$ is of order one, the radiation mean free path is approximately equal to one in units of ξ. The disturbance velocity is plotted as a function of time at a series of fixed positions. The τ and ξ scales are broken at several points to produce a better visualization of the entire flow field. Between breaks the scales are linear and would have their origins at the intersection of the τ and ξ axes if continued back to the origin. Notice that the scale size is quite different in the separate regions.

At a point located a small distance from the wall, the velocity takes a sudden jump at a time equal to that required for a signal traveling at the isentropic sound speed to reach the point. In other words, initially, the disturbance is a unit step propagating at the isentropic speed. The step dies out exponentially with increasing distance from the wall and is replaced by a smooth transition from zero at $\tau=0$ to a value of 1.0 at $\tau\rightarrow\infty$. In this process the center of the dis-

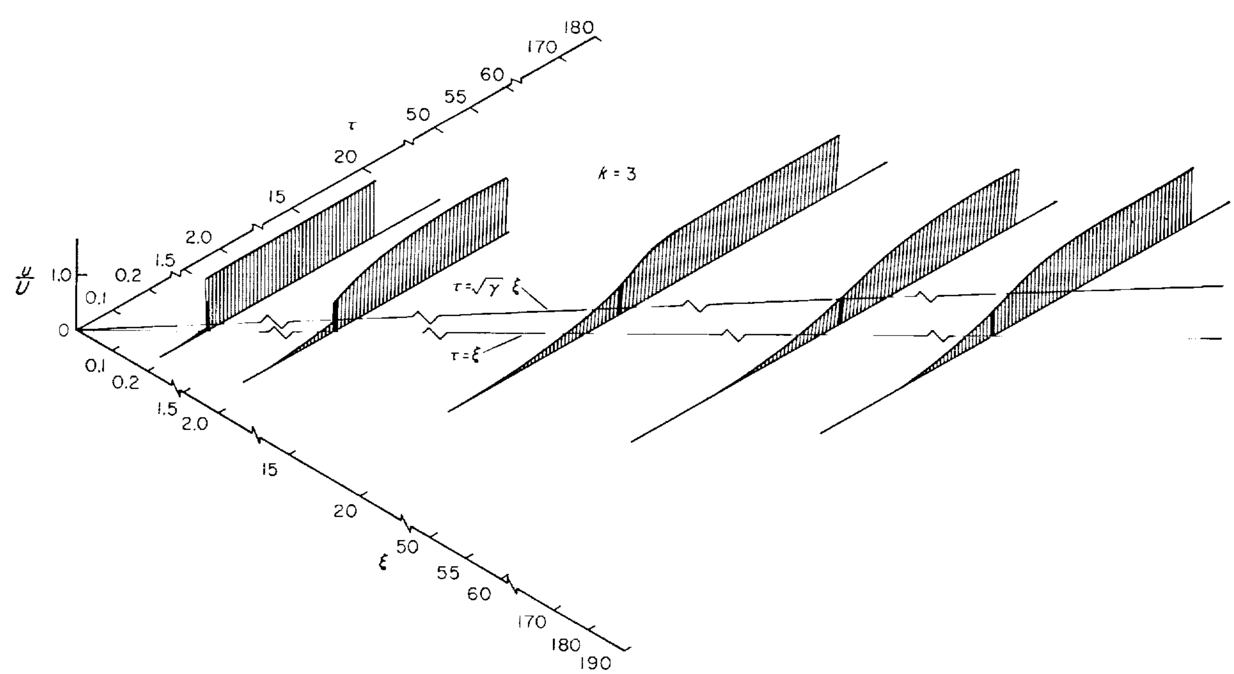


FIGURE 3.—Disturbance velocity response to impulsive motion of piston ($\kappa=3.0$).

turbance shifts toward the path of an isothermal signal ($\tau=\sqrt{\gamma}\xi$). Eventually, at a large distance from the wall, the center of the disturbance shifts back to the path of an isentropic signal at $\tau=\xi$. The center of the disturbance is taken to be the point where the velocity has reached ½ of its final value when plotted as a function of τ for fixed ξ. This point is indicated in each subplot of figure 3 by a heavy vertical line under the curve.

At small and intermediate distances from the wall, the response is similar to that for a gas in chemical or vibrational nonequilibrium (see refs. 31 and 35). For that case the disturbance is initially a unit step propagating at the frozen speed of sound. The step dies out exponentially, the wave front becomes dispersed, and its center shifts to the slower equilibrium speed. The subsequent behavior differs from that for a radiating gas in that the disturbance continues to travel at the slower speed rather than switching back to the starting value. There is a further difference between the two cases. For a radiating gas, there are no characteristics corresponding to a finite velocity. Consequently, a precursor extends ahead of the path of an isentropic signal at $\tau=\xi$ (see fig. 3). No such effect occurs for chemical or vibrational nonequilibrium because there is a characteristic corresponding to the frozen sound speed, and hence no disturbance ahead of this line.

A qualitative physical explanation can be given for the response of a radiating gas to the impulsive motion as follows: Referring again to figure 3, at small distance from the wall the wave front is compact such that its width is small compared to the radiation mean free path (equal to 1 in units of ξ). As a result, the radiative heat transfer *within* the wave front is negligible. The disturbance is then governed by the isentropic condition and travels at isentropic speed. As it progresses, the wave front becomes dispersed by the small, but nonzero, radiative heat-transfer process. When the width of the wave becomes comparable to the radiation mean free path, heat transfer can readily occur within the front. This tends to hold the temperature constant, depending on the intensity of radiation. At high initial temperature of the gas (large k), the temperature is held essentially constant within the wave front, and the disturbance travels at the isothermal speed.

At large distances from the wall, the wave front becomes so dispresed that the radiative heat transfer is impeded as a result of reabsorption relatively near the point of emission. Thus, when the width of the front becomes large compared to the radiation mean free path, the adiabatic condition applies, and the disturbance travels at the isentropic speed.

There is another useful qualitative explanation for the behavior depicted in figure 3. These results can be related to the properties of the solution for sinusoidal motion of the wall, given in figure 2, as follows. At small ξ, the wave front is compact. Consequently, at a fixed point near the wall, the gas velocity varies rapidly with time. A Fourier analysis of such a rapid variation would indicate a preponderance of high frequencies. In figure 2 it can be seen that at high frequencies the speed of the modified classical wave v_1 is equal to the isentropic speed of sound a_0. This accounts for the initial propagation of the compression wave at the isentropic speed. At intermediate values of ξ, where we see in figure 3 that the disturbance is partially dispersed, the Fourier analysis would show the peak amplitude to be at intermediate frequency. Figure 2 indicates an approach to the isothermal speed at intermediate frequency, in agreement with the shift to a slower speed in figure 3. When the wave front is further dispersed, at large ξ, the resulting low frequencies lead to a prediction of the observed return to the isentropic speed. Obviously, there are gaps in the foregoing explanation, if it is not supported by other information. But such a point of view may be useful in other problems involving a Fourier transform that is difficult to invert.

An understanding of the response to an impulsive wall motion is not complete without consideration of the effect of varying the radiation parameter k. This quantity is a measure of the intensity of radiation. It is defined in equation (76), which is $k=16\sqrt{\frac{(\gamma+1)}{2}}(\gamma-1)\left(\frac{m}{n}\right)\sigma T_0^3/\gamma R\rho_0 a_0$. At small values of k, the response should become that associated with classical acoustic theory, namely a unit step in velocity propagating unchanged at the isentropic speed. At large values of k, one would again expect the classical result, but with the isentropic speed replaced with the isothermal speed, since the large radiative heat transfer associated with small temperature differences

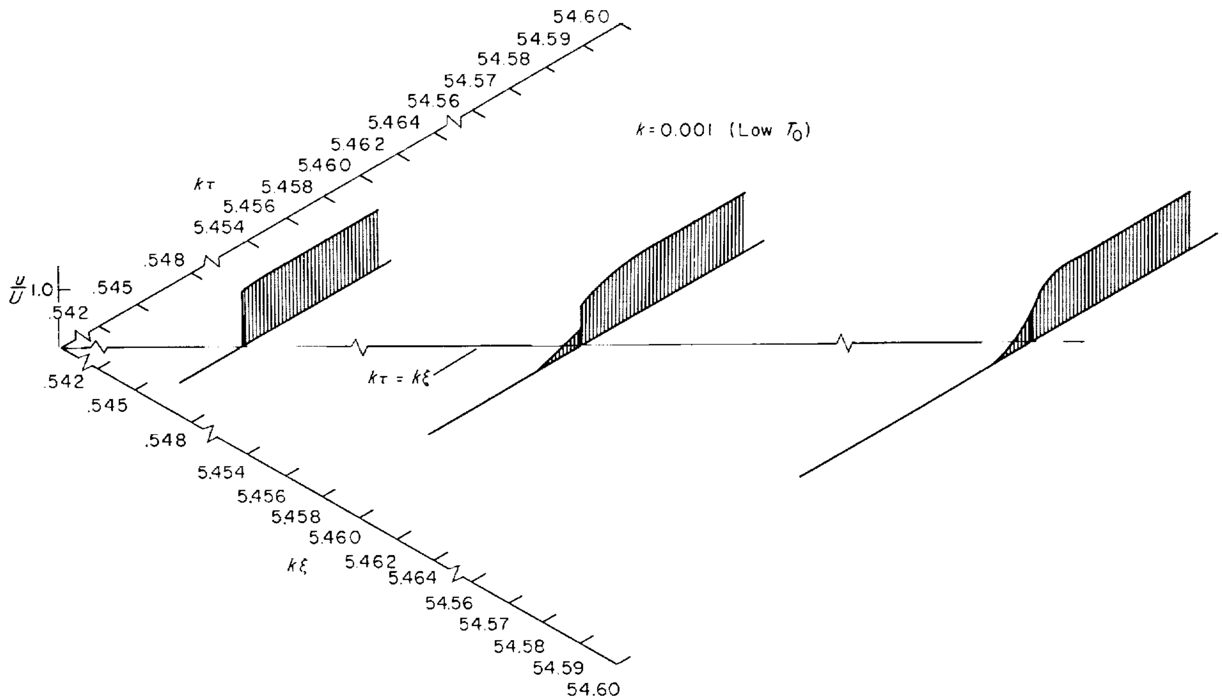


FIGURE 4.—Disturbance velocity response to impulsive motion of piston ($\kappa=0.001$).

would hold the temperature constant. These expectations are realized in a sense, but there are singular perturbation effects at both small and large k resulting in a somewhat more complicated picture. Such effects could be predicted, at least in part, from the results given in reference 1 (response to sinusoidal motion of the wall). There it is shown that, for large k, the classical result of undamped sine waves traveling at the isothermal speed is obtained, except for very high or very low frequencies (see fig. 3 of ref. 1).

At small k, equation (102) indicates that the quantity b becomes 1.0, and using equation (101), equation (100) can be written as

$$\left.\frac{u_A(\tau,\xi)}{U}\right|_{k\to 0}=\frac{1}{2}\,(1-\exp\,[-(\sqrt{\gamma}-1)k\xi])$$
$$\left(\operatorname{erf}\left\{\frac{\tau-\xi}{2[(\sqrt{\gamma}-1)k\xi]^{1/2}}\right\}+\operatorname{erf}\left\{\frac{\tau+\xi}{2[(\sqrt{\gamma}-1)k\xi]^{1/2}}\right\}\right)$$
$$+\frac{1}{2}\exp\,[-(\sqrt{\gamma}-1)k\xi]\left(1+\frac{\tau-\xi}{|\tau-\xi|}\right)\quad(103)$$

At all values of ξ except large values comparable to $\frac{1}{k}$, the quantity $\exp\,[-(\sqrt{\gamma}-1)k\xi]$ is equal to 1.0 so that equation (103) represents a unit step propagating at the isentropic speed (the path of an isentropic signal is at $\tau=\xi$). This is the classical result which is expected for $k=0$ when radiation effects are absent. However, at large values of ξ comparable to $1/k$, the step dies out exponentially and is replaced by the error function variation. The distance from the wall at which this occurs goes to infinity as k goes to zero. At such distances, the transition in velocity still occurs in a region narrow compared to the distance, and the transition is centered on the path of an isentropic signal. This result is plotted in figure 4. Since the unit step propagates unchanged until the distance from the wall becomes large of order k^{-1}, the region where it does change is brought into focus by plotting the velocity as a function of $k\tau$ and $k\xi$ rather than τ and ξ. Two other features in figure 4 require explanation. The width of the region where the velocity transition occurs depends on the value of k and is narrow compared to the distance from the wall. Therefore, a specific small value of k equal to 0.001 is used. Also the scales are broken in several places with different scale sizes in each region.

For small k we see in figure 4 that the disturbance remains centered about the path of an isen-

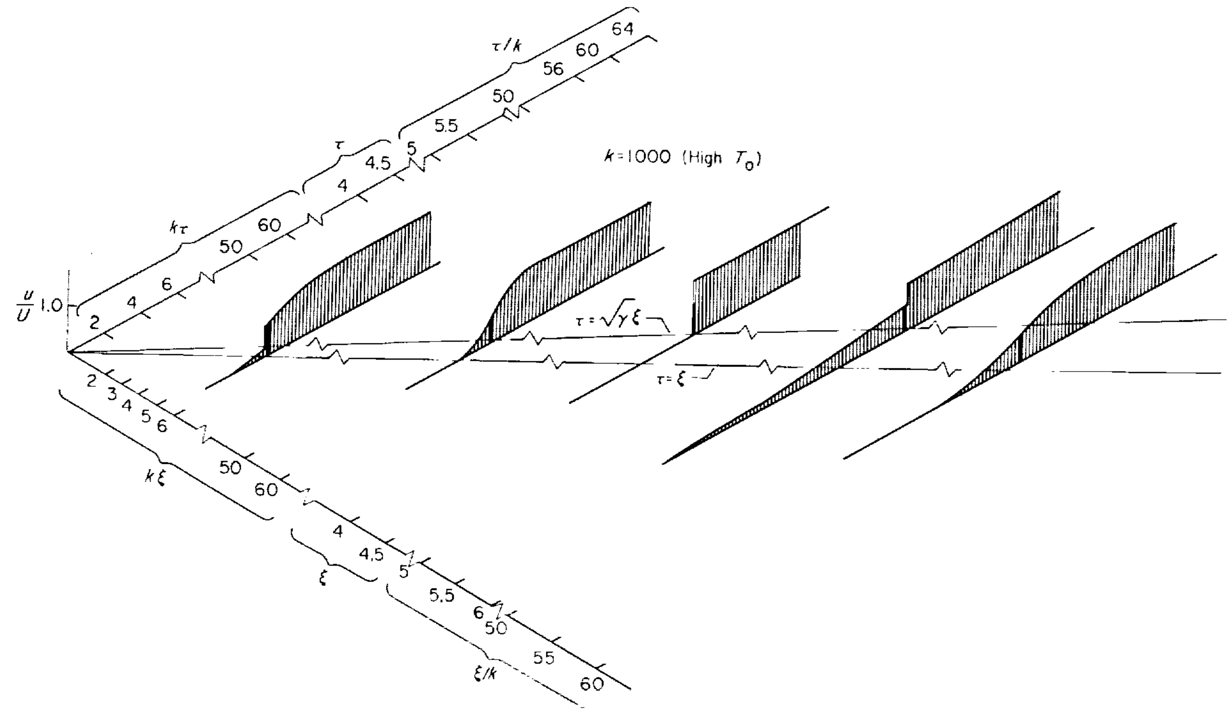


FIGURE 5.—Disturbance velocity response to impulsive motion of piston ($\kappa=1000$).

tropic signal ($k\tau=k\xi$). The radiation merely causes a dispersion so that the unit step is eventually replaced by a smooth transition.

At large values of k (high temperature), equation (102) indicates that the quantity b is equal to k and equation (100) becomes

$$\frac{u_A(\tau,\xi)}{U}\bigg|_{k\to\infty}=\frac{1}{2}\left\{1-\exp\left[-(\sqrt{\gamma}-1)\frac{\xi}{k}\right]\right\}$$
$$\left(\operatorname{erf}\left\{\frac{\tau-\xi}{2[(\sqrt{\gamma}-1)k\xi]^{1/2}}\right\}+\operatorname{erf}\left\{\frac{\tau+\xi}{2[(\sqrt{\gamma}-1)k\xi]^{1/2}}\right\}\right)$$
$$+\frac{1}{2}\exp\left[-(\sqrt{\gamma}-1)k\xi\right]\left(1+\frac{\tau-\xi}{|\tau-\xi|}\right)$$
$$+\frac{1}{2}\left\{\exp\left[-(\sqrt{\gamma}-1)\frac{\xi}{k}\right]-\exp[-(\sqrt{\gamma}-1)k\xi]\right\}$$
$$\left(\operatorname{erf}\left\{\frac{k(\tau-\sqrt{\gamma}\xi)}{2[(\sqrt{\gamma}-1)k\xi]^{1/2}}\right\}\right.$$
$$\left.+\operatorname{erf}\left\{\frac{k(\tau+\sqrt{\gamma}\xi)}{2[(\sqrt{\gamma}-1)k\xi]^{1/2}}\right\}\right)\quad(104)$$

Three ranges of distance from the wall must be considered in this case. All three ranges are shown in figure 5. Here the scales are broken in several places and are linear between breaks as in figure 3. A value of $k=1000$ was used to compute the points, but the graphs would remain similar for larger values. There is a boundary layer in which ξ is small of order $\frac{1}{k}$ where the appropriate inner variables are $k\tau$ and $k\xi$, as shown in figure 5. In this layer $\frac{\xi}{k}$ is small compared to 1.0 and the quantity $\exp\left[-(\sqrt{\gamma}-1)\frac{\xi}{k}\right]$ becomes 1.0. Then equation (104) simplifies to

$$\frac{u_A(\tau,\xi)}{U}\bigg|_{\substack{k\to\infty\\ k\xi\text{ fixed}\\ k\tau\text{ fixed}}}=\frac{1}{2}\{1-\exp\left[-(\sqrt{\gamma}-1)k\xi\right]\}$$
$$\left(\operatorname{erf}\left\{\frac{k\tau-\sqrt{\gamma}k\xi}{2[(\sqrt{\gamma}-1)k\xi]^{1/2}}\right\}+\operatorname{erf}\left\{\frac{k\tau+\sqrt{\gamma}k\xi}{2[(\sqrt{\gamma}-1)k\xi]^{1/2}}\right\}\right)$$
$$+\frac{1}{2}\exp\left[-(\sqrt{\gamma}-1)k\xi\right]\left(1+\frac{k\tau-k\xi}{|k\tau-k\xi|}\right)\quad(105)$$

Within the boundary layer this result includes a step which propagates at isentropic speed and dies out exponentially. The error function terms represent a smooth transition centered about the path of an isothermal signal ($k\tau=\sqrt{\gamma}k\xi$). When $k\xi$ becomes large compared to 1.0, the latter variation replaces the step completely and we have

$$\left.\frac{u_A}{U}\right|_{\substack{k\to\infty\\ k\xi \text{ fixed}\\ k\tau \text{ fixed}\\ k\xi\to\infty}}=\frac{1}{2}\left(1+\operatorname{erf}\left\{\frac{k\tau-\sqrt{\gamma}k\xi}{2[(\sqrt{\gamma}-1)k\xi]^{1/2}}\right\}\right) \quad (106)$$

This function represents a transition from zero to one in a narrow region centered about $k\tau=\sqrt{\gamma}k\xi$.

In the next range of distance from the wall, the appropriate variables are τ and ξ. These are taken to be large compared to $\frac{1}{k}$, but small compared to k. In that case $\exp\ [-(\sqrt{\gamma}-1)\xi/k]=1.0$, $\exp\ [-(\sqrt{\gamma}-1)k\xi]=0$, and $\operatorname{erf}\ \left[\frac{k\tau+\sqrt{\gamma}k\xi}{2(\sqrt{\gamma}-1)k\xi}\right]=1.0$.

Equation (105) then becomes

$$\left.\frac{u_A}{U}\right|_{\substack{k\to\infty\\ \xi \text{ fixed}\\ \tau \text{ fixed}}}=\frac{1}{2}\left(1+\operatorname{erf}\left\{\frac{\sqrt{k}(\tau-\sqrt{\gamma}\xi)}{2[(\sqrt{\gamma}-1)\xi]^{1/2}}\right\}\right) \quad (107)$$

By cancellation of $\sqrt{k}$ from the numerator and denominator of equation (106), it is seen that the result at the outer edge of the boundary layer matches equation (107), which applies outside the boundary layer. In terms of the physical variables τ and ξ, equation (107) represents a unit step propagating at the isothermal speed. At least it is a step in the limit as k goes to infinity, since the error function structure then becomes compressed into an infinitesimal width. Thus, except for very small ξ of order $\frac{1}{k}$ and very large ξ of order k (see below), the expected classical result of a unit step with isothermal speed is obtained. Then the step which appears at $\tau=\sqrt{\gamma}\xi$ in the middle and outer regions in figure 5 has an error function structure, not resolved in the graphs.

The last range of distance from the wall, of interest for the case of large k, comes into focus in terms of the variables $\frac{\tau}{k}$ and $\frac{\xi}{k}$. When these are taken to be large compared to $\frac{1}{k^2}$, equation (104) simplifies to the expression

$$\left.\frac{u_A(\tau,\xi)}{U}\right|_{\substack{k\to\infty\\ \xi/k \text{ fixed}\\ \tau/k \text{ fixed}}}=\frac{1}{2}\left\{1-\exp\left[-(\sqrt{\gamma}-1)\frac{\xi}{k}\right]\right\}\left[1+\operatorname{erf}\left\{\frac{\left(\frac{\tau}{k}\right)-\left(\frac{\xi}{k}\right)}{2\left[(\sqrt{\gamma}-1)\frac{\xi}{k}\right]^{\frac{1}{2}}}\right\}\right]$$
$$+\frac{1}{2}\exp\left[-(\sqrt{\gamma}-1)\frac{\xi}{k}\right]\left(1+\operatorname{erf}\left\{\frac{k\left(\frac{\tau}{k}-\sqrt{\gamma}\frac{\xi}{k}\right)}{2\left[(\sqrt{\gamma}-1)\frac{\xi}{k}\right]^{\frac{1}{2}}}\right\}\right) \quad (108)$$

At $\xi/k<<1.0$, this reduces to equation (107), and hence represents a unit step propagating at the isothermal speed. As ξ/k increases, the step dies out exponentially and is replaced by the error function variation indicated in the first term. In this process the precursor, extending ahead of the line $\tau=\xi$, reappears (see fig. 5). At values of $\xi/k>>1.0$, equation (108) becomes

$$\left.\frac{u_A(\tau,\xi)}{U}\right|_{\substack{k\to\infty\\ \xi/k \text{ fixed}\\ \tau/k \text{ fixed}\\ \xi/k\to\infty}}=\frac{1}{2}\left(1+\operatorname{erf}\left\{\frac{\frac{\tau}{k}-\frac{\xi}{k}}{2\left[(\sqrt{\gamma}-1)\frac{\xi}{k}\right]^{\frac{1}{2}}}\right\}\right) \quad (109)$$

This result yields the asymptotic behavior at large ξ/k. The transition of the velocity disturbance from zero to one starts ahead of the path of an isentropic signal ($\tau/k=\xi/k$) and is completed in a width of order $[(\sqrt{\gamma}-1)\xi/k]^{\frac{1}{2}}\approx[(\sqrt{\gamma}-1)\tau/k]^{\frac{1}{2}}$. Thus the width of the wave front grows parabolically at large distances from the wall; whereas the separation between the paths of isothermal and isentropic signals increases at a faster linear rate.

In retrospect it can be seen that the behavior of the disturbance for large k is qualitatively similar to that for the intermediate value of $k=3.0$, described earlier and pictured in figure 3. But for large k, the switch from isentropic toward isothermal speed of the wave front occurs at smaller distances from the wall, is more complete, and persists to a larger distance from the wall.

Numerical investigations have not been made of the pressure and temperature fields. However, the behavior of these quantities can be described qualitatively using the results from the limiting cases discussed earlier and in appendix E. Since the results for large k are similar to those for intermediate values, only large k will be considered here.

Referring to figure 5, at values of $k\xi<<1.0$ (i.e., near the wall) the velocity undergoes a unit step at $k\tau=k\xi$ and thereafter remains constant. The dimensionless perturbation pressure p'/ρ_0a_0U also takes a unit step at this point, but does not remain constant thereafter. From equations (F.1)–(F.3) and (F.14), it can be seen that in a time of order $k\tau=1.0$, the pressure at the wall drops to a value of $1/\sqrt{\gamma}$. It remains at this value until a large time of order $\tau=k$, when it slowly returns to a value of 1.0 and thereafter remains constant.

At a point located outside the boundary layer such that $k^{-1}<<\xi<<k$ the velocity undergoes an error function variation from zero to one in a narrow region near $\tau=\sqrt{\gamma}\xi$. The perturbation pressure follows a similar variation from zero to $1/\sqrt{\gamma}$ in this region. But rather than remaining constant thereafter as does the velocity, the perturbation pressure eventually rises to 1.0 at a time of order $\tau=k$.

At large distance from the wall, where ξ is large compared to k, the pressure variation follows that of the velocity. This is an error function variation from zero to one in a relatively narrow region near $\tau=\xi$, and no further change. In all three regions the final value of the perturbation pressure is 1.0, which is the same as that which would occur in the absence of radiation ($k=0$).

Equations (F.1)–(F.3) and (F.15) can be used to find the behavior of the temperature for the case of large k. At the wall the dimensionless perturbation temperature RT'/a_0U undergoes a step at $\tau=0$ similar to that taken by the velocity and pressure, but the amplitude is equal to $(\gamma-1)/\gamma$. This is the same variation as would occur in the absense of radiation. However, in a time of order $k\tau=1.0$, the perturbation temperature drops asymptotically to zero by radiation to the wall, which is held at constant temperature. It should perhaps be reiterated that, in the present inviscid approximation, the temperature of the gas adjacent to the wall need not be equal to the wall temperature at all times because of the presence of an optically thin thermal boundary layer.

At a fixed value of ξ of order one, the perturbation temperature (as a function of τ) remains zero in the neighborhood of $\tau=\sqrt{\gamma}\ \xi$ where the velocity and pressure rise. It subsequently rises slowly and falls again to zero at some time greater than $\tau=k$.

At large distances from the wall ($\xi>>k$), the temperature participates, along with the pressure and velocity, in an isentropic variation near $\tau=\xi$. Thus the perturbation temperature, RT'/a_0U, rises to a value of $(\gamma-1)/\gamma$. In the subsequent time, while the pressure and velocity remain constant, the temperature slowly returns to its initial value of zero in a time greater than $\tau=k$. The final perturbation temperature in all three regions is zero owing to radiation to the wall, which is held at constant temperature.

In reference 1, a solution corresponding to sinusoidal temperature variations of a fixed wall is discussed, as well as the results for sinusoidal motion of a wall at constant temperature. The problem with impulsive time dependence, corresponding to the former, would be that of an impulsive temperature variation of a fixed wall. This is an interesting case. Although no complete study of the problem has been made, a qualitative description of the results to be expected is given in appendix H.

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APPENDIX A

TABLE OF SYMBOLS

a	quantity equal to $(k\beta/\gamma)/[1+2\beta^2/(\gamma+1)]$
a_0	isentropic speed of sound in undisturbed gas
a_1, a_2	positive finite nonzero constants defined in equations (E.8) and (E.9)
A	complex constant defined in equation (C.3)
A_1, A_2	complex wave amplitudes defined in equations (77) and (78)
b	imaginary part of one root of $\nu^2-ik+1=0$ (see eq. (F.19)) also used as a parameter equal to $\alpha_0 a_0/\omega$ in appendix C
b_1, b_2, b_3	positive finite nonzero constants defined in equations (E.8), (E.9), and (E.13)
B	complex constant defined in equation (C.4)
B_ν	Planck function, see equation (8)
c	velocity of light
c_1, c_2	complex constants containing wave speeds and damping constants of modified-classical wave and radiation-induced wave, respectively
C	constant equal to $8\sigma T_0^3\alpha_0$
C_j	complex wave amplitudes
erf (η)	error function equal to $(2/\sqrt{\pi})\int_0^\eta e^{-s^2}ds$
$E_n(\eta)$	integro-exponential function equal to $\int_0^1 e^{-(\eta/\mu)}\mu^{(n-2)}d\mu$
$f(t, x)$	arbitrary function
$\bar{f}(\omega, x)$	Fourier transform of $f(t, x)$
$f(t)$	dimensionless wall velocity defined in equation (B.8)
$F(\eta)$	attenuation function defined in equation (26)
$g(t)$	dimensionless perturbation wall temperature defined in equation (B.9)
h	Planck's constant, also enthalpy per unit mass
$H(y)$	complex quantity defined in equation (C.2)
Im	imaginary part
I_1, I_2, I_3	integrals defined in equation (E.7)
k	Boltzmann's constant, also used as a quantity equal to $16\sqrt{(\gamma+1)/2}(m/n)(\gamma-1)\sigma T_0^3/(\gamma R\rho_0 a_0)$
K	equal to $16(m/n)(\gamma-1)\sigma T_0^3/(R\rho_0 a_0)$
m, n	quantities defined in equations (34), (37), and (38)
$0(\epsilon)$	quantity of order ϵ
p	gas pressure
p'	perturbation pressure
p_0, p_1, p_2	components of pressure in expansion (see appendix B)
Q	net heat absorbed per unit volume and time due to radiation
Q_ν	net heat absorbed per unit volume, time and frequency interval due to radiation
$(Q/\alpha)_1, (Q/\alpha)_2$	components of (Q/α) in expansion (see appendix B)
r	variable equal to $k\nu$
R	gas constant
Re	real part of
t	time
T	gas temperature
T_0	undisturbed gas temperature
T'	perturbation temperature
T_1, T_2	components of temperature in expansion (see appendix B)
$\tilde{T}$	perturbation temperature at $x=\tilde{x}$
$T_w(t)$	wall temperature
$\overline{T}_w'(\omega)$	Fourier transform of perturbation wall temperature
u, u'	gas velocity
u_1, u_2	components of gas velocity in expansion (see appendix B)
$u_w(t)$	wall velocity

$\bar{u}_w(\omega)$ Fourier transform of wall velocity

u_A approximate gas velocity in impulsive piston problem

u_N remaining part of gas velocity after subtraction of approximate part (evaluated numerically)

U constant wall velocity at $t>0$ in impulsive-piston problem

x Cartesian coordinate

x_w coordinate at the position of the wall

X variable equal to $(\sqrt{\gamma}-1)k\xi$

α Planck mean radiation absorption coefficient defined in equation (12)

α_0 value of α in the undisturbed gas

α_1, α_2 components of α in expansion (see appendix B)

α_ν frequency dependent radiation absorption coefficient

β parameter equal to $n\alpha_0 a_0/\omega$

γ ratio of specific heats

ϵ a number small compared to one

η variable equal to $\int_{x_w(t)}^{x} \alpha d\hat{x}$

$\tilde{\eta}$ dummy variable of integration

η_ν variable equal to $\int_0^\eta \frac{\alpha_\nu}{\alpha} d\hat{\eta}$

$\eta_{\nu 0}$, $\eta_{\nu 1}$ components of η_ν in expansion (see eqs. (B.14) and (B.15))

θ variable equal to t (used to facilitate transformation)

Θ value of wall temperature at $t>0$

κ parameter equal to $8(\gamma-1)\sigma T_0^3/R\rho_0 a_0$

ν frequency of electromagnetic radiation

ξ variable equal to $\sqrt{2/(\gamma+1)}\, n\alpha_0 x$

ρ gas density

ρ_0 density of undisturbed gas

ρ_1, ρ_2 components of density in expansion (see appendix B)

σ Stefan Boltzmann constant equal to 5.673×10^{-5} erg cm^{-2} deg^{-4} sec^{-1}

τ variable equal to $\sqrt{2/(\gamma+1)}\, n\alpha_0 a_0 t$

φ velocity potential

ω radian frequency of oscillation in oscillating piston problem

SUBSCRIPTS AND SPECIAL SYMBOLS

0 denotes the undisturbed gas condition (as in ρ_0)

1, 2 components of expansion used in appendix B, for example, ρ_1, ρ_2

$'$ denotes perturbation quantity (as in $\rho'=\rho-\rho_0$)

APPENDIX B

SECOND-ORDER EQUATIONS

To help establish that the linearization used in the text is imbedded in a rational successive-approximation procedure, the second-order equations will be derived in this appendix. The small-disturbance expansion could be carried out in a number of different ways. As stated in the text, nonuniformities can be partially avoided by transforming to the variables t, η in place of t, x. This procedure will be used here.

The one-dimensional unsteady inviscid-flow equations are

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x}=0 \tag{B.1}$$

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\frac{\partial p}{\partial x}=0 \tag{B.2}$$

$$\rho\frac{\partial h}{\partial t}+\rho u\frac{\partial h}{\partial x}-\frac{\partial p}{\partial t}-u\frac{\partial p}{\partial x}=Q \tag{B.3}$$

(see ref. 1). These are supplemented by equations (6)–(17) of the text, which define Q, and the equations of state for a perfect gas, which are

$$h=\frac{\gamma}{\gamma-1}RT \tag{B.4}$$

$$T=\frac{1}{R}\frac{p}{\rho} \tag{B.5}$$

The boundary conditions are

$$u[t,x_w(t)]=u_w(t) \tag{B.6}$$

$$u(t,x) \qquad \text{finite at } x\rightarrow 0 \tag{B.7}$$

The wall velocity $u_w(t)$ and wall temperature $T_w(t)$ are taken to be given functions of t ($T_w(t)$ appears in the expression for Q).

We wish to expand the flow quantities about a zero value of an appropriate parameter that is a measure of the magnitude of the disturbance. For this purpose, $u_w(t)$ and $T_w(t)$ can be expressed as

$$u_w(t)=\epsilon\frac{a_0}{\gamma}f(t) \tag{B.8}$$

$$T_w(t)=T_0+\epsilon T_0 g(t) \tag{B.9}$$

where $f(t)$ and $g(t)$ are given functions of t with maximum values of order one. Then ϵ is a dimensionless parameter that goes to zero in the limit of a vanishingly small disturbance. It will be assumed that all quantities can be expanded in powers of ϵ at least to second order, for example,

$$T=T_0+\epsilon T_1+\epsilon^2 T_2+\ .\ .\ . \tag{B.10}$$

We will first expand the quantity (Q/α) to second order in ϵ, and return later to the other equations. The variable (Q/α) is chosen, rather than Q, to promote simplifications which will become apparent later. Equations (6) and (17) can be combined and written as

$$\frac{Q}{\alpha}=2\pi\int_0^\infty\left(\frac{\alpha_\nu}{\alpha}\right)\left\{[B_\nu(T_w)-B_\nu(T)|_{\eta=0}]E_2(\eta_\nu)-\int_0^\eta\frac{dB_\nu(\tilde{T})}{d\tilde{T}}E_2(\eta_\nu-\tilde{\eta}_\nu)\frac{\partial\tilde{T}}{\partial\tilde{\eta}}d\tilde{\eta}+\int_\eta^\infty\frac{dB_\nu(\tilde{T})}{d\tilde{T}}E_2(\tilde{\eta}_\nu-\eta_\nu)\frac{\partial\tilde{T}}{\partial\tilde{\eta}}d\tilde{\eta}\right\}d\nu \tag{B.11}$$

where

$$\eta_\nu=\int_0^\eta\frac{\alpha_\nu}{\alpha}d\hat{\eta} \tag{15}$$

The integral over ν can be carried further by expansion of equations (B.11) and (15) in powers of ϵ. For this purpose, the following expansions are needed

$$\frac{\alpha_\nu}{\alpha}=\left(\frac{\alpha_\nu}{\alpha}\right)_0+\epsilon\left(\frac{\alpha_\nu}{\alpha}\right)_1+0(\epsilon^2) \tag{B.12}$$

$$\eta_\nu=\eta_{\nu_0}+\epsilon\eta_{\nu_1}+0(\epsilon^2) \tag{B.13}$$

By substitution of equation (B.12) into (15), and noting that $\left(\frac{\alpha_\nu}{\alpha}\right)_0$ is constant, it can be established that

$$\eta_{\nu_0}=\left(\frac{\alpha_\nu}{\alpha}\right)_0\eta \tag{B.14}$$

$$\eta_{\nu_1}=\int_0^\eta\left(\frac{\alpha_\nu}{\alpha}\right)_1 d\hat{\eta} \tag{B.15}$$

Power series expansion of the E_2 function yields

$$E_2(\eta_\nu)=E_2(\eta_{\nu_0}+\epsilon\eta_{\nu_1}+\ldots)$$
$$=E_2(\eta_{\nu_0})+\frac{dE_2(\eta_{\nu_0})}{d\eta_{\nu_0}}\epsilon\eta_{\nu_1}+0(\epsilon^2)$$

The relation $dE_2(\eta)/d\eta=-E_1(\eta)$ can be used to write this expression as

$$E_2(\eta_\nu)=E_2(\eta_{\nu_0})-\epsilon E_1(\eta_{\nu_0})\eta_{\nu_1}+0(\epsilon^2) \tag{B.16}$$

Since the E_1 function is logarithmically singular at a zero value of its argument, the radius of convergence of the power series expansion of the E_2 function goes to zero as η_{ν_0} goes to zero. However, this does not invalidate the expansion, since $\epsilon\eta_{\nu_1}$ goes to zero at the same rate as does the radius of convergence. In other words, equation (B.16) remains valid at all values of η. This can be seen by noting that as η goes to zero, where $E_1(\eta_{\nu_0})$ is logarithmically singular, equation (B.15) shows that η_{ν_1} goes to zero algebraically. Then the combination $E_1(\eta_{\nu_0})\eta_{\nu_1}$ remains finite and, in fact, goes to zero as η goes to zero. The same would be true of all higher-order terms in equation (B.16).

Using equations (B.12)–(B.16), the quantity $\frac{\alpha_\nu}{\alpha}E_2(\eta_\nu)$ in the first term of the integrand of equation (B.11) can be written as

$$\frac{\alpha_\nu}{\alpha}E_2(\eta_\nu)=\left(\frac{\alpha_\nu}{\alpha}\right)_0 E_2\left[\left(\frac{\alpha_\nu}{\alpha}\right)_0\eta\right]$$
$$+\epsilon\left\{\left(\frac{\alpha_\nu}{\alpha}\right)_1 E_2\left[\left(\frac{\alpha_\nu}{\alpha}\right)_0\eta\right]-\left(\frac{\alpha_\nu}{\alpha}\right)_0 E_1\left[\left(\frac{\alpha_\nu}{\alpha}\right)_0\eta\right]\right.$$
$$\left.\int_0^\eta\left(\frac{\alpha_\nu}{\alpha}\right)_1 d\hat{\eta}\right\}+0(\epsilon^2) \tag{B.17}$$

For the purpose of expressing (Q/α) in terms of T, we will continue to concentrate on the first term of the integrand of equation (B.11). By power series expansion of the function $B_\nu(T)$, given in equation (8), and using equation (B.10), we find

$$B_\nu(T)=B_\nu(T_0)+\epsilon\frac{dB_\nu(T_0)}{dT_0}T_1$$
$$+\epsilon^2\left[\frac{dB_\nu(T_0)}{dT_0}T_2+\frac{1}{2}\frac{d^2B_\nu(T_0)}{dT_0^2}T_1^2\right]+\ldots \tag{B.18}$$

From this and equation (B.9) it follows that

$$B_\nu(T_w)-B_\nu(T)|_{\eta=0}=\epsilon\frac{dB_\nu(T_0)}{dT_0}[T_0g(t)-T_1|_{\eta=0}]$$
$$+\epsilon^2\left\{-\frac{dB_\nu(T_0)}{dT_0}T_2|_{\eta=0}+\frac{1}{2}\frac{d^2B_\nu(T_0)}{dT_0^2}\right.$$
$$\left.[T_0^2g^2(t)-T_1^2|_{\eta=0}]\right\}+\ldots \tag{B.19}$$

With the assumption of a Boltzmann distribution of states, the quantity α_ν/α is a function only of temperature (for a fixed value of ν), since it is proportional to the mass absorption coefficient. This is the reason for choosing to expand (Q/α) rather than Q. Expansion of α_ν/α in powers of $(T-T_0)$ and use of equations (B.10) and (B.12) leads to the result

$$\left(\frac{\alpha_\nu}{\alpha}\right)_1=\frac{d\frac{\alpha_\nu}{\alpha}(T_0)}{dT_0}T_1 \tag{B.20}$$

Then, since T_0 is constant, it follows that

$$\int_0^\eta\left(\frac{\alpha_\nu}{\alpha}\right)_1 d\hat{\eta}=\frac{d\frac{\alpha_\nu}{\alpha}(T_0)}{dT_0}\int_0^\eta T_1 d\hat{\eta} \tag{B.21}$$

Use of equations (B.12), (B.17), (B.20), and (B.21) and a similar expansion of the integral terms of equation (B.11) leads to the results

$$Q/\alpha=\epsilon(Q/\alpha)_1+\epsilon^2(Q/\alpha)_2+\ldots \tag{B.22}$$

$$(Q/\alpha)_1=\frac{C}{\alpha_0}\left\{[T_0g(t)-T_1|_{\eta=0}]F(\eta)\right.$$
$$\left.-\int_0^\infty\frac{(\eta-\tilde{\eta})}{|\eta-\tilde{\eta}|}F(|\eta-\tilde{\eta}|)\frac{\partial T_1}{\partial\tilde{\eta}}d\tilde{\eta}\right\} \tag{B.23}$$

$$
\begin{aligned}
(Q/\alpha)_2=\frac{C}{\alpha_0}\Big\{ & -T_2|_{\eta=0}F(\eta) \\
& -\int_0^\infty \frac{(\eta-\tilde{\eta})}{|\eta-\tilde{\eta}|} F(|\eta-\tilde{\eta}|) \frac{\partial T_2}{\partial \tilde{\eta}} d\tilde{\eta}\Big\} \\
& +\frac{C_I}{\alpha_0}\Big\{[T_0^2 g^2(t)-T_1^2|_{\eta=0}]F_I(\eta) \\
& -\int_0^\infty \frac{(\eta-\tilde{\eta})}{|\eta-\tilde{\eta}|} F_I(|\eta-\tilde{\eta}|) \frac{\partial T_1^2}{\partial \tilde{\eta}} d\tilde{\eta}\Big\} \\
& +\frac{C_{II}}{\alpha_0}\frac{\partial}{\partial \eta}\Big\{[T_0 g(t)-T_1|_{\eta=0}]\int_0^\eta T_1 d\hat{\eta} F_{II}(\eta) \\
-\int_0^\infty \frac{(\eta-\tilde{\eta})}{|\eta-\tilde{\eta}|} & F_{II}(|\eta-\tilde{\eta}|)\int_{\tilde{\eta}}^\eta T_1 d\hat{\eta}\, \frac{\partial T_1}{\partial \tilde{\eta}} d\tilde{\eta}\Big\}
\end{aligned} \tag{B.24}
$$

where the functions F, F_I and F_{II} are defined by the relations

$$\frac{C}{\alpha_0} F(\eta)=2\pi\int_0^\infty \left(\frac{\alpha_\nu}{\alpha}\right)_0 \frac{dB_\nu(T_0)}{dT_0} E_2\left[\left(\frac{\alpha_\nu}{\alpha}\right)_0 \eta\right] d\nu \tag{B.25}$$

$$\frac{C_I}{\alpha_0} F_I(\eta)=\pi\int_0^\infty \left(\frac{\alpha_\nu}{\alpha}\right)_0 \frac{d^2B_\nu(T_0)}{dT_0^2} E_2\left[\left(\frac{\alpha_\nu}{\alpha}\right)_0 \eta\right] d\nu \tag{B.26}$$

$$\frac{C_{II}}{\alpha_0} F_{II}(\eta)=2\pi\int_0^\infty \frac{d\frac{\alpha_\nu}{\alpha}(T_0)}{dT_0}\frac{dB_\nu(T_0)}{dT_0} E_2\left[\left(\frac{\alpha_\nu}{\alpha}\right)_0 \eta\right] d\nu \tag{B.27}$$

The constants C, C_I, and C_{II} are arbitrary. In the text, C was chosen such that for a grey gas $F(\eta)$ becomes equal to $E_2(\eta)$. A similar choice here leads to

$$C=8\sigma T_0^3\alpha_0 \tag{25}$$

$$C_I=12\sigma T_0^3\alpha_0 \tag{B.28}$$

$$C_{II}=8\sigma T_0^3\alpha_0 \tag{B.29}$$

Then for a grey gas $F_I(\eta)$ also becomes equal to $E_2(\eta)$. The function $F_{II}(\eta)$ introduces the effect of temperature dependence of the mass absorption coefficient into the equation. In the case of a grey gas, this function is zero.

In the foregoing procedure there are no obvious difficulties that would prevent an extension of the expansion to higher order in ϵ. It can be shown that an attempt to expand $E_2(\eta_\nu)$ using x as the basic variable instead of η will fail because, in general, x is not zero when η_ν is zero (see eq. (10)). A new variable $\hat{x}=x-x_w(t)$ could probably be used, but the second-order part of (Q/α) would contain additional terms depending on the density. Altogether, η is probably the most convenient variable, when the expansion in ϵ is extended beyond the linear approximation.

It can be seen that the transformed coordinate system t, η is not an inertial reference frame. This complicates the expansion of the continuity, momentum, and energy equations. However, it will be seen that this complication is similar to one which occurs in classical acoustic theory also, and can be treated. To facilitate the transformation procedure, t will be replaced by θ. Then we wish to express equations (B.1)–(B.3) in terms of the variables θ, η defined by

$$\theta=t \tag{B.30}$$

$$\eta=\int_{x_w(t)}^x \alpha(t,\hat{x})d\hat{x} \tag{13}$$

The transformation of derivatives is, in general,

$$\frac{\partial}{\partial t}=\frac{\partial\theta}{\partial t}\frac{\partial}{\partial\theta}+\frac{\partial\eta}{\partial t}\frac{\partial}{\partial\eta} \tag{B.31}$$

$$\frac{\partial}{\partial x}=\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}+\frac{\partial\eta}{\partial x}\frac{\partial}{\partial\eta} \tag{B.32}$$

These can be partially evaluated using equations (B.30) and (13) to obtain

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial\theta}+\frac{\partial\eta}{\partial t}\frac{\partial}{\partial\eta} \tag{B.33}$$

$$\frac{\partial}{\partial x}=\alpha\frac{\partial}{\partial\eta} \tag{B.34}$$

The quantity $\frac{\partial\eta}{\partial t}$ can also be evaluated in terms of t and x using equation (13). However, in the transformed equations, $\frac{\partial\eta}{\partial t}$ must be expressed as a function of θ and η. For that purpose, equations (B.30) and (13) must be inverted to obtain expressions for t and x in terms of θ and η. Since α is a function of temperature and density, the inversion will depend on the condition of the flow field, in general. However, it turns out that with an expansion about $\epsilon=0$, the inversion can be found to any desired order in ϵ by a "boot strap" type procedure.

Differentiation of equations (B.30) and (13) yields

$$d\theta = dt \tag{B.35}$$

$$d\eta = \frac{\partial\eta}{\partial t}dt + \alpha dx \tag{B.36}$$

where

$$\frac{\partial\eta}{\partial t} = -\alpha_w \frac{dx_w}{dt} + \int_{x_w(t)}^{x} \frac{\partial\alpha(t,\hat{x})}{\partial t}\,d\hat{x} \tag{B.37}$$

Inversion of equations (B.35) and (B.36) leads to

$$t = \theta \tag{B.30}$$

$$dx = \frac{1}{\alpha}d\eta - \frac{1}{\alpha}\frac{\partial\eta}{\partial t}d\theta \tag{B.38}$$

Using equations (B.8), (B.30), (13), and (B.38), equation (B.37) can be written

$$\frac{\partial\eta}{\partial t} = -\epsilon\alpha_w \frac{a_0}{\gamma}f(\theta) + \int_0^{\eta} \frac{1}{\alpha}\frac{\partial\alpha(t,\hat{x})}{\partial t}\,d\hat{\eta} \tag{B.39}$$

To express this as a function of θ and η, a transformation of $\frac{\partial\alpha}{\partial t}$ according to equation (B.33) is needed. But $\frac{\partial\eta}{\partial t}$ itself appears in that equation. The process can be carried out, nevertheless, because, for evaluation of $\frac{\partial\eta}{\partial t}$ to a given order in ϵ, $\frac{\partial}{\partial t}$ is only needed to the next lowest order. For this purpose, an expansion of α is required as follows

$$\alpha = \alpha_0 + \epsilon\alpha_1 + 0(\epsilon^2) \tag{B.40}$$

Substitution in equation (B.33) and using the fact that α_0 is constant leads to

$$\frac{\partial\alpha}{\partial t} = \epsilon\frac{\partial\alpha_1}{\partial\theta} + \epsilon\frac{\partial\eta}{\partial t}\frac{\partial\alpha_1}{\partial\eta} + 0(\epsilon^2) \tag{B.41}$$

When this is substituted in equation (B.39), it is seen that $\frac{\partial\eta}{\partial t}$ is of order ϵ. Hence

$$\frac{\partial\alpha}{\partial t} = \epsilon\frac{\partial\alpha_1}{\partial\theta} + 0(\epsilon^2) \tag{B.42}$$

and it follows that

$$\frac{\partial\eta}{\partial t} = -\epsilon\alpha_0\frac{a_0}{\gamma}f(\theta) + \frac{\epsilon}{\alpha_0}\int_0^{\eta}\frac{\partial\alpha_1}{\partial\theta}\,d\hat{\eta} + 0(\epsilon^2) \tag{B.43}$$

This procedure could be carried to higher order, but for the second-order results the derivatives need only be transformed to order ϵ, since the quantities on which they operate are of order ϵ.

Expanding the flow quantities in powers of ϵ and using equations (B.33), (B.34), and (B.43) leads to an expression containing ϵ and ϵ^2 terms. Equating the coefficient of the ϵ term to zero and replacing θ with t yields

$$\frac{\partial\rho_1}{\partial t} + \alpha_0\rho_0\frac{\partial u_1}{\partial\eta} = 0 \tag{B.44}$$

$$\rho_0\frac{\partial u_1}{\partial t} + \alpha_0\frac{\partial p_1}{\partial\eta} = 0 \tag{B.45}$$

$$\rho_0\frac{\partial h_1}{\partial t} - \frac{\partial p_1}{\partial t} - \alpha_0\left(\frac{Q}{\alpha}\right)_1 = 0 \tag{B.46}$$

Similarly, equating the coefficient of the ϵ^2 term to zero leads to the expressions

$$\frac{\partial\rho_2}{\partial t} + \rho_0\alpha_0\frac{\partial u_2}{\partial\eta} = -\left[-\frac{a_0\alpha_0}{\gamma}f(t) + \frac{1}{\alpha_0}\int_0^{\eta}\frac{\partial\alpha_1}{\partial t}d\hat{\eta} + \alpha_0 u_1\right]\frac{\partial\rho_1}{\partial\eta} - \alpha_0\rho_1\frac{\partial u_1}{\partial\eta} + \frac{\alpha_1}{\alpha_0}\frac{\partial\rho_1}{\partial t} \tag{B.47}$$

$$\rho_0\frac{\partial u_2}{\partial t} + \alpha_0\frac{\partial p_2}{\partial\eta} = -\rho_0\left[-\frac{a_0\alpha_0}{\gamma}f(t) + \frac{1}{\alpha_0}\int_0^{\eta}\frac{\partial\alpha_1}{\partial t}d\hat{\eta} + \alpha_0 u_1\right]\frac{\partial u_1}{\partial\eta} - \rho_1\frac{\partial u_1}{\partial t} + \frac{\alpha_1}{\alpha_0}\frac{\partial u_1}{\partial t} \tag{B.48}$$

$$\rho_0\frac{\partial h_2}{\partial t} - \frac{\partial p_2}{\partial t} - \alpha_0\left(\frac{Q}{\alpha}\right)_2 = -\left[-\frac{a_0\alpha_0}{\gamma}f(t) + \frac{1}{\alpha_0}\int_0^{\eta}\frac{\partial\alpha_1}{\partial t}d\hat{\eta} + \alpha_0 u_1\right]\left(\rho_0\frac{\partial h_1}{\partial\eta} - \frac{\partial p_1}{\partial\eta}\right) - \rho_1\frac{\partial h_1}{\partial t} + \frac{\alpha_1}{\alpha_0}\left(\rho_0\frac{\partial h_1}{\partial t} - \frac{\partial p_1}{\partial t}\right) \tag{B.49}$$

The quantity α_1 appearing here should be expressed in terms of the other variables. This can be done by expanding the quantity (α/ρ) in powers of $(T-T_0)$. It can be seen in equation (12) that (α/ρ) is a function of temperature alone, since the mass absorption coefficient (α_ν/ρ) is a function of temperature alone in the assumed absence of nonequilibrium processes other than radiation. It is found that α_1 is given by

$$\alpha_1=\alpha_0\frac{\rho_1}{\rho_0}+\rho_0\frac{d\frac{\alpha}{\rho}(T_0)}{dT_0}T_1 \tag{B.50}$$

The expansion of the equations of state is straightforward and need not be discussed. Finally, using equations (B.6), B.7), and (B.8), the boundary conditions can be written as

$$u_1|_{\eta=0}=\frac{a_0}{\gamma}f(t) \tag{B.51}$$

$$u_2|_{\eta=0}=0 \tag{B.52}$$

and all quantities must be finite at $\eta\rightarrow\infty$. The foregoing expansion procedure could evidently be extended to any desired order in ϵ.

When solutions of the foregoing equations in terms of t and η are found, a parametric representation would result from also expressing x as a function of t and η. Such a relation follows from an integration of equation (B.38) using equation (B.43). If θ is replaced with t and equation (B.8) used, the result is

$$x=\frac{\eta}{\alpha_0}+x_w(t)-\frac{\epsilon}{\alpha_0^2}\int_0^\eta[2\alpha_1(t,\hat{\eta})-\alpha_1(0,\hat{\eta})]d\hat{\eta}+0(\epsilon^2) \tag{B.53}$$

The linearized results given in the text can now be derived. Expansion of equations (B.4) and (B.5) in powers of ϵ yields

$$h_1=\frac{\gamma}{\gamma-1}RT_1 \tag{B.54}$$

$$T_1=\frac{1}{R}\frac{p_1}{p_0}-\frac{1}{\gamma R}\frac{a_0^2}{\rho_0}\rho_1 \tag{B.55}$$

Equation (B.45) is satisfied identically if a velocity potential is defined by the relations

$$u_1=\alpha_0\frac{\partial\varphi_1}{\partial\eta} \tag{B.56}$$

$$p_1=-\rho_0\frac{\partial\varphi_1}{\partial t} \tag{B.57}$$

Substitution of (B.56) into (B.44) leads to

$$\frac{\partial\rho_1}{\partial t}=-\rho_0\alpha_0^2\frac{\partial^2\varphi_1}{\partial\eta^2} \tag{B.58}$$

Differentiation of equation (B.55) with respect to t and substitution of equations (B.57) and (B.58) yields

$$\frac{\partial T_1}{\partial t}=-\frac{1}{R}\left(\frac{\partial^2\varphi_1}{\partial t^2}-\frac{a_0^2}{\gamma}\alpha_0^2\frac{\partial^2\varphi_1}{\partial\eta^2}\right) \tag{B.59}$$

where use has been made of the perfect gas relation for the isentropic speed of sound

$$a_0^2=\gamma RT_0 \tag{B.60}$$

Substitution of the foregoing relations into equation (B.46) leads to

$$\frac{\partial^2\varphi_1}{\partial t^2}-a_0^2\alpha_0^2\frac{\partial^2\varphi_1}{\partial\eta^2}=-(\gamma-1)\frac{\alpha_0}{\rho_0}\left(\frac{Q}{\alpha}\right)_1 \tag{B.61}$$

It can be seen that $\left(\frac{Q}{\alpha}\right)_1=\frac{1}{\alpha_0}Q_1+0(\epsilon)$, so that the last equation can be written as

$$\frac{\partial^2\varphi_1}{\partial t^2}-a_0^2\alpha_0^2\frac{\partial^2\varphi_1}{\partial\eta^2}=-(\gamma-1)\frac{Q_1}{\rho_0} \tag{B.62}$$

To obtain equations (1)–(5) of the text from these equations two further steps are required. First, the primed quantities in the text are perturbation quantities so that, to lowest order, each is ϵ times the corresponding quantity with subscript 1. Since the ϵ would appear as a factor in every term, it would cancel. Secondly, if $x_w(t)$ is neglected in equation (B.53), we see that x is equal to $\frac{\eta}{\alpha_0}$ to lowest order. With this substitution in equations (B.56)–(B.59) and (B.62), we obtain equations (1)–(5) of the text. Alternatively, instead of neglecting $x_w(t)$, we can measure x from the wall. That is, we can interpret the variable appearing in the text as a new variable $\hat{x}$, with $\hat{x}=x-x_w(t)$, so that $\hat{x}=\frac{\eta}{\alpha_0}$ to lowest order. It will be seen in the following discussion that the second alternative is essential in the present problem.

Let us consider an expansion about $\epsilon=0$ with x as variable in place of η. As in any acoustic theory, such a development would include a transfer of the boundary conditions from $x=x_w$ to $x=0$; that is,

$$u(t,x_w)=u(t,0)+\left.\frac{\partial u(t,x)}{\partial x}\right|_{x=0}x_w+0(x_w^2) \tag{B.63}$$

It can be seen by integration of equation (B.8) that x_w is formally of order ϵ. In classical acoustic theory this fact is used to arrive at the conclusion that the boundary condition for u_1 is

$$u_1(t,x)|_{x=0}=\frac{a_0}{\gamma}f(t)$$

When $x_w(t)$ is large, as in the impulsive-piston problem at large t, it is customary to replace x with $x-x_w(t)$ in the solution. It can be seen from equation (B.53) that, to lowest order, this is equivalent to using η as a variable instead of x. For the case of a radiating gas, there is an important additional consideration. In appendices C and D it is shown that the exact solution (for sinusoidal boundary conditions) of the foregoing linearized equations is not analytic at the wall. It follows that the boundary conditions cannot be translated according to equation (B.63), even for small x_w. Therefore it is essential to use η as a variable rather than x in this problem. Although x cannot be used, the variable $\hat{x}=x-x_w(t)$ could be, since it would not be necessary to transfer the boundary conditions in that case. This is the basis for the statement made in the text, that the equations obtained by neglecting x_w are correct to lowest order if x is measured from the wall, rather than from a fixed origin.

One other type of nonuniformity appears in the present linearization, as well as in classical acoustic theory. A discrepancy appears at large distances from the source of a disturbance owing to cumulative nonlinear effects. This can be illustrated by considering the impulsive-piston problem, wherein a compression wave travels outward from the wall. In the foregoing discussion, it is concluded that, in the first approximation, x should be measured from the wall. For disturbances in the rear of the compression wave this is correct, since they are indeed moving into a gas that is at rest relative to the wall. On the other hand, the foremost part of the wave front is moving into a fixed gas. Thus the wave velocity is in error by a small amount at the front of the disturbance, an effect which can lead to a large error in the relative positions of elements of the disturbance in the course of a movement over a large distance. It can be seen that similar errors will result from evaluating the wave speeds at the temperature of the undisturbed gas, rather than at the correct local temperature. These effects have been treated by a coordinate stretching procedure (refs. 25 and 26). In this technique, the linearized results are not discarded in favor of a new approach. The independent variables are replaced in the solution by new variables which are functions of the original variables. It is assumed that this procedure can be carried out in the present problem, but the matter will not be further investigated here.

APPENDIX C

INVESTIGATION OF EXPONENTIAL APPROXIMATION OF ATTENUATION FACTOR

In reference 1 and in the present work, the attenuation factor $F(\eta)$ appearing as the kernel of the integro-differential equation (30), is approximated by an exponential according to equation (34). In this appendix the validity of the approximation is investigated in the case of a grey gas, for which

$$F(\eta)=E_2(\eta)=\int_0^1 e^{-\eta/\mu}d\mu \tag{C.1}$$

It is found that the approximation does not yield a uniformly valid approximation for the gradient of temperature at the wall. The same is true for higher derivatives of the other physical quantities. However, no nonuniformities are found in the approximations for the physical quantities themselves near the wall. At large distance from the wall two kinds of error can occur as follows: (1) In any acoustic theory, cumulative nonlinear effects appear in the evaluation of a flow field far from the source of a disturbance, even in the lowest approximation. If the disturbance is sufficiently small at large distances, these effects may not be important. When they are important, the linearized results must be corrected to obtain a uniformly valid approximation (see refs. 25 and 26). (2) If an approximate solution is used, rather than an exact linear result, there may be additional cumulative effects which will cause error in the prediction of the flow field far from the source of disturbance. Again, such effects may not be important if the disturbance attenuates sufficiently. These matters will not be investigated here. In the following, an attempt will be made to find any other possible sources of error which may result from the exponential approximation of the attenuation factor. For this purpose we will concentrate on the solution for the response of a grey gas to sinusoidal boundary conditions. In that case it is expedient to express the potential and the boundary conditions in the following forms

$$\varphi=\frac{RT_0}{\omega}\,\mathrm{Re}\,[H(y)e^{i\omega t}] \tag{C.2}$$

$$\left.\frac{\partial\varphi}{\partial x}\right|_{x=0}=\frac{RT_0}{a_0}\,\mathrm{Re}\,[Ae^{i\omega t}] \tag{C.3}$$

$$\frac{dT_w}{dt}=\omega T_0\,\mathrm{Re}\,[Be^{i\omega t}] \tag{C.4}$$

$$\varphi(t,\infty)=\frac{RT_0}{\omega}\,\mathrm{Re}\,[H(\infty)e^{i\omega t}]=\text{finite quantity} \tag{C.5}$$

where

$$y=\omega x/a_0 \tag{C.6}$$

(see ref. 1). The quantities A and B are dimensionless complex constants, assumed to be specified. The magnitudes of A and B must be small compared to one, in conformity with the requirement of small disturbances imposed in the linearization.

Substitution of equation (C.2) into equation (30) yields

$$\begin{aligned}H(y)+H''(y)=-i\kappa b\Bigg\{&\Bigg[B-H(0)-\frac{1}{\gamma}H''(0)\Bigg]E_2(by)-\int_0^y E_2[b(y-\bar{y})]\Bigg[H'(\bar{y})\\&+\frac{1}{\gamma}H'''(\bar{y})\Bigg]d\bar{y}+\int_y^\infty E_2[b(\bar{y}-y)]\Bigg[H'(\bar{y})\\&+\frac{1}{\gamma}H'''(\bar{y})\Bigg]d\bar{y}\Bigg\}\end{aligned} \tag{C.7}$$

where

$$\kappa=8(\gamma-1)\sigma T_0^3/R\rho_0 a_0 \tag{C.8}$$

and

$$b=\alpha_0 a_0/\omega \tag{C.9}$$

The boundary conditions on the complex dimensionless quantity $H(y)$ are found from equations

(C.2) to (C.5) and are

$$H'(0)=A \tag{C.10}$$

$$H(\infty)=\text{finite quantity} \tag{C.11}$$

In reference 1, approximate solutions of equation (C.7) are found by replacing the E_2 function with an exponential. Here we will consider an approximation of the form

$$E_2(\eta)\approx\sum_{l=1}^{L} m_l e^{-n_l\eta} \tag{C.12}$$

Solutions can then be found in the form

$$H(y)=\sum_{j=1}^{J} C_j e^{c_j y} \tag{C.13}$$

Substitution of equations (C.12) and (C.13) into equations (C.7) and (C.10) and equating the coefficient of each resulting exponential term to zero yields the relations

$$1+c_j^2-i2\kappa b\left(1+\frac{c_j^2}{\gamma}\right)\sum_{l=1}^{L} m_l\frac{c_j^2}{c_j^2-n_l^2b^2}=0 \qquad \text{for all } j \tag{C.14}$$

$$\sum_{j=1}^{J}\left(1+\frac{c_j^2}{\gamma}\right)\left(\frac{n_l b}{n_l b+c_j}\right)C_j=B \qquad \text{for all } l \tag{C.15}$$

$$\sum_{j=1}^{J} c_j C_j=A \tag{C.16}$$

Specializing to the case $L=2$, equation (C.14) can be written as

$$1+c_j^2-iK\beta\left(1+\frac{c_j^2}{\gamma}\right)\left(\frac{c_j^2}{c_j^2-\beta^2}\right)=iK\beta\frac{m_2}{m_1}\frac{\left(1+\frac{c_j^2}{\gamma}\right)c_j^2}{c_j^2-\left(\frac{n_2}{n_1}\beta\right)^2} \tag{C.17}$$

where

$$K=2\frac{m_1}{n_1}\kappa \tag{C.18}$$

and

$$\beta=n_1 b \tag{C.19}$$

Equation (C.17) is a sixth-degree algebraic equation in c_j. The six roots are then the quantities c_j in the solution represented by equation (C.13). The integro-differential equation for $H(y)$ can be converted to a purely differential equation of sixth order. It follows that all solutions of the integro-differential equation are contained in the general solution, equation (C.13), with $J=6$. Since the characteristic equation (C.17) contains c_j only in the combination c_j^2, half of the roots will have positive real parts. The corresponding values of C_j must be set equal to zero to satisfy the boundary condition, equation (C.11). Taking the first three roots to be those with negative real part, the J in equation (C.13) can then be set equal to three, and equations (C.15) and (C.16) can be solved for the amplitudes C_1, C_2, C_3 in terms of the known quantities A, B, c_1, c_2, c_3.

If m_2/m_1 is small compared with one, the roots of equation (C.17) can be found by an expansion about m_2/m_1 equal to zero. The lowest order part of the result is that obtained by setting the right side of equation (C.17) equal to zero. This is the characteristic equation used in reference 1 and in the text of the present work. It has previously been shown that this equation can be solved to a good approximation by an expansion for small values of $\sqrt{\gamma}-1$. This expansion will be used here, since it greatly facilitates the manipulations. The results from the double expansion are

$$c_1=-i\left\{1+(\sqrt{\gamma}-1)\frac{ia}{1-ia}-\frac{m_2}{m_1}(\sqrt{\gamma}-1)\frac{K\beta}{\left[1+\left(\frac{n_2}{n_1}\beta\right)^2\right][1-ia]}+\text{higher order in }(\sqrt{\gamma}-1)\text{ and } m_2/m_1\right\} \tag{C.20}$$

$$c_2=-\frac{\beta}{\sqrt{1-iK\beta\gamma}}\left\{1+(\sqrt{\gamma}-1)\frac{ia}{1-ia}+\frac{m_2}{m_1}(\sqrt{\gamma}-1)\frac{K(1-iK\beta)}{\beta\left[1+\left(\frac{n_2}{n_1}\beta\right)^2\right][1-ia]}+\text{higher order in }(\sqrt{\gamma}-1)\text{ and } m_2/m_1\right\} \tag{C.21}$$

$$c_3=-\left(\frac{n_2\beta}{n_1}\right)\left\{1-\frac{m_2}{2m_1}\frac{iK\beta[(n_2/n_1)^2-1]}{(n_2/n_1)^2-1-iK\beta(n_2/n_1)^2}+\text{higher order in }(\sqrt{\gamma}-1)\text{ and } m_2/m_1\right\} \tag{C.22}$$

where

$$a=\frac{K\beta/\gamma}{1+\frac{2}{\gamma+1}\beta^2} \tag{C.23}$$

Specializing to the case of constant wall temperature ($B=0$), equations (C.15) and (C.16) can be solved with the aid of the double expansion to obtain

$$C_1=iA+0(\sqrt{\gamma}-1)+0\left[(\sqrt{\gamma}-1)\frac{m_2}{m_1}\right] \tag{C.24}$$

$$C_2=0(\sqrt{\gamma}-1)+0\left[(\sqrt{\gamma}-1)\frac{m_2}{m_1}\right] \tag{C.25}$$

$$C_3=-(\sqrt{\gamma}-1)\frac{m_2}{m_1}A\frac{K\beta\left[\left(\frac{n_2}{n_1}\right)^2-1\right]}{\left[(1-iK\beta)\left(\frac{n_2}{n_1}\right)^2-1\right][1-ia]}\frac{\frac{n_2}{n_1}\beta}{\left(\frac{n_2}{n_1}\beta-i\right)\left[\left(\frac{n_2}{n_1}\beta\right)^2-1\right]}$$

$$+\text{ higher order in }(\sqrt{\gamma}-1)\text{ and }m_2/m_1 \tag{C.26}$$

From these results it can be seen that if m_2/m_1 is small compared to one, the values of c_1, c_2, C_1, and C_2 will differ only by small amounts from their values at $m_2/m_1=0$ for all values of K and β. Such a change in the wave speeds could cause a nonuniformity at large x because it is a cumulative effect. For the response of a grey gas to an impulsive motion of the wall, however, it can be shown that such a nonuniformity does not occur. This is so because the $0\left[\frac{m_2}{m_1}(\sqrt{\gamma}-1)\right]$ corrections to the wave speeds go to zero at the large values of β involved in the evaluation of the solution at large x. In other words, the component waves with appreciable wave speed discrepancies are sufficiently damped that their amplitudes become negligible at large x, where their positions are given inaccurately by the single-exponential approximation of the attenuation factor. Possible nonuniformities of this nature should be considered in applications involving nongrey gases or wall boundary conditions different than those used here.

The amplitude C_3 goes to zero when m_2/m_1 is zero and will cause only a small change in $H(y)$ when m_2/m_1 differs from zero by a small amount. However, since c_3 is equal to $(-n_2\beta/n_1)$ to lowest order, derivatives of $C_3\exp(c_3y)$ can become large at $y=0$ when $n_2\beta/n_1$ is large. Substitution of equations (C.20) to (C.26) into equation (C.13) and differentiation three times with respect to y leads to the expression

$$H'''(0)=-iA-(\sqrt{\gamma}-1)\frac{m_2}{m_1}A\frac{K\beta\left[\left(\frac{n_2}{n_1}\right)^2-1\right]}{\left[(1-iK\beta)\left(\frac{n_2}{n_1}\right)^2-1\right][1-ia]}\frac{\left(\frac{n_2}{n_1}\beta\right)^4}{\left(\frac{n_2}{n_1}\beta-i\right)\left[\left(\frac{n_2}{n_1}\beta\right)^2-1\right]}$$

$$+0(\sqrt{\gamma}-1)+0\left[(\sqrt{\gamma}-1)\left(\frac{m_2}{m_1}\right)^2\right] \tag{C.27}$$

The neglected terms are small compared to those retained for all values of the parameters. The term proportional to m_2/m_1 is small compared to the first for small m_2/m_1 at all values of the parameters, except when $n_2\beta/n_1$ is large compared to one. For that case equation (C.27) can be simplified to

$$H'''(0)=-iA-(\sqrt{\gamma}-1)\frac{m_2}{m_1}A\frac{K\beta\left[\left(\frac{n_2}{n_1}\right)^2-1\right]}{\left[(1-iK\beta)\left(\frac{n_2}{n_1}\right)^2-1\right][1-ia]}\times\frac{n_2}{n_1}\beta+\ldots \tag{C.28}$$

Using equations (C.2), (2), (3), and (4), it can be seen that the first derivative of temperature, second derivative of velocity, and third derivative of pressure (evaluated at the wall) each have terms proportional to $H'''(0)$. Hence these and higher derivatives of T, u, and p become singular at the wall as $(n_2\beta/n_1)$ approaches infinity. However, the region of nonuniformity becomes exponentially small, since the C_3 term involved is multiplied by $\exp[-(n_2\beta/n_1)y]$. An effect of this type is predicted in reference 1. The basis for the prediction can be seen from equation (C.7), in which the first term on the right is proportional to $E_2(by)$. The first derivative of the E_2 function is logarithmically singular at a zero value of its argument, whereas the first derivative of the integral terms of equation (C.7) is not. It follows that the only remaining part of the equation, the quantity $H(y)+H''(y)$, must match the singular behavior; that is, $H'''(y)$ must be infinite at $y=0$. Further details on the form of $H'''(y)$ near $y=0$ are given in appendix D, where a procedure for obtaining an exact solution is discussed.

To determine at which values of β and y the foregoing nonuniformity becomes important, specific values of m and n are needed. Values of m_1, m_2, n_1, and n_2 have been found which satisfy the following requirements:

(1)

$$(m_1e^{-n_1\eta}+m_2e^{-n_2\eta})_{\eta=0}=E_2(0)=1.0$$

(2)

$$\int_0^\infty (m_1e^{-n_1\eta}+m_2e^{-n_2\eta})e^{-n_2\eta}d\eta=\int_0^\infty E_2(\eta)e^{-n_2\eta}d\eta$$

(3)

$$\int_0^\infty (m_1e^{-n_1\eta}+m_2e^{-n_2\eta})\eta d\eta=\int_0^\infty E_2(\eta)\eta d\eta=\frac{1}{3}$$

(4)

Least squares fit of $m_1e^{-n_1\eta}+m_2e^{-n_2\eta}$ to $E_2(\eta)$

The first two of these conditions were chosen to obtain a good fit of the approximating function to $E_2(\eta)$ at small values of η. The third requirement insures a correct result in the Rosseland limit of strong absorption. The last condition is self-explanatory. The resulting values of m and n are

$$m_1=0.745$$
$$n_1=1.532$$
$$m_2=0.255$$
$$n_2=20.$$

The second term in equation (C.28) has its largest value when $K\beta$ and a are of order 1. For a value of $\gamma=7/5$, and the foregoing values for m and n, the second term begins to exceed the first at about $\beta=1.0$. The region of nonuniformity at that point is confined to values of y less than about 0.1 and is confined to y less than $0.1/\beta$ for larger β.

Three types of possible nonuniformities arising from the present approximation procedure have been disclosed in the investigations contained in this and the preceding appendix. The most serious are probably those that can occur at large distance from the wall. In the classical acoustic theory, nonuniformities of this type have been treated by a coordinate stretching process. The details of the procedure for a similar program in the present problem are far from obvious.

APPENDIX D

EXACT SOLUTION FOR A GREY GAS

In this appendix the validity of the exponential approximation of the attenuation factor will be further investigated for the case of a grey gas and oscillatory boundary conditions. Equations (C.1) through (C.7) of appendix C are the exact relations for this case. Equation (C.7) is a linear integro-differential equation similar to that appearing in the Milne problem (isotropic scattering of radiation or of slow neutrons; see, e.g., ref. 30). A method previously employed for solution of the Milne problem could therefore be used here. This method, which is exact, utilizes the Fourier transform plus the Wiener-Hopf technique for factoring the transform. In reference 32 a solution based on the Wiener-Hopf technique is given for a problem even more nearly analogous to the present one than the Milne problem. An investigation of equation (C.7) by this method would be desirable, but will not be made here. Instead, an alternative procedure will be used which yields information on the properties of the solution at small and large values of y. No numerical examples will be given.

If equation (C.13) is substituted into (C.7), and the integral expression equation (C.1) substituted for the E_2 function, it is found that the resulting relation can be partially evaluated by an interchange of the order of integration. A solution would then be obtained by setting the coefficients of each exponential term equal to zero, but there is an additional part containing an integral of a function of μ times $\exp(-by/\mu)$ which cannot be zero. This suggests that the solution is of the form

$$H(y)=\sum_{j=1}^{J} C_j e^{c_j y}+\int_0^1 G(\theta)e^{-by/\theta}d\theta \qquad \text{(D.1)}$$

When the procedure just described is applied using this expression, the result is

$$\sum_{j=1}^{J}\left\{1+c_j^2-i\kappa b\left(1+\frac{c_j^2}{\gamma}\right)\left[2+\frac{b}{c_j}\ln\left(\frac{b-c_j}{b+c_j}\right)\right]\right\}C_je^{c_jy}$$
$$+\int_0^1\left\{1+\frac{b^2}{\theta^2}-i\kappa b\left(1+\frac{b^2}{\gamma\theta^2}\right)\left[2-\theta\ln\left(\frac{1+\theta}{1-\theta}\right)\right]\right\}G(\theta)e^{-by/\theta}d\theta=i\kappa b\left[-B\int_0^1 e^{-by/\mu}d\mu\right.$$
$$+\sum_{j=1}^{J}\left(1+\frac{c_j^2}{\gamma}\right)C_j\int_0^1\left(\frac{b}{b+c_j\mu}\right)e^{-by/\mu}d\mu$$
$$\left.+\int_{\theta=0}^1\left(1+\frac{b^2}{\gamma\theta^2}\right)G(\theta)\int_{\mu=0}^1\left(\frac{\theta}{\theta-\mu}\right)e^{-by/\mu}d\mu d\theta\right] \qquad \text{(D.2)}$$

In evaluating the integrals over μ on the left of this equation, the principal part was taken of a singular integral. The same must be done in evaluating the singular integral on the right side, since the two singularities cancel each other.

If the dummy variable of integration, θ, on the left side is changed to μ, equation (D.2) can be written as

$$\sum_{j=1}^{J}\left\{1+c_j^2-i\kappa b\left(1+\frac{c_j^2}{\gamma}\right)\left[2+\frac{b}{c_j}\ln\left(\frac{b-c_j}{b+c_j}\right)\right]\right\}C_je^{c_jy}$$
$$=\int_{\mu=0}^1\left(-\left\{1+\frac{b^2}{\mu^2}-i\kappa b\left(1+\frac{b^2}{\gamma\mu^2}\right)\left[2-\mu\ln\left(\frac{1+\mu}{1-\mu}\right)\right]\right\}G(\mu)+i\kappa b\left[-B\right.\right.$$
$$+\sum_{j=1}^{J}\left(1+\frac{c_j^2}{\gamma}\right)C_j\left(\frac{b}{b+c_j\mu}\right)+\int_{\theta=0}^1$$
$$\left.\left.\left(1+\frac{b^2}{\gamma\theta^2}\right)G(\theta)\left(\frac{\theta}{\theta-\mu}\right)d\theta\right]\right)e^{-by/\mu}d\mu \qquad \text{(D.3)}$$

This equation will be satisfied if the following relations hold

$$1+c_j^2-i\kappa b\left(1+\frac{c_j^2}{\gamma}\right)\left[2+\frac{b}{c_j}\ln\left(\frac{b-c_j}{b+c_j}\right)\right]=0 \quad \text{for all } j \qquad \text{(D.4)}$$

$$\left\{1+\frac{b^2}{\mu^2}-i\kappa b\left(1+\frac{b^2}{\gamma\mu^2}\right)\left[2-\mu\ln\left(\frac{1+\mu}{1-\mu}\right)\right]\right\}G(\mu)$$
$$=i\kappa b\left[-B+\sum_{j=1}^{J}\left(1+\frac{c_j^2}{\gamma}\right)C_j\left(\frac{b}{b+c_j\mu}\right)+\int_{\theta=0}^{1}\left(1+\frac{b^2}{\gamma\theta^2}\right)G(\theta)\left(\frac{\theta}{\theta-\mu}\right)d\theta\right] \quad \text{(D.5)}$$

Equation (D.4) is a transcendental equation and has an infinite number of roots. However, by appealing to a principle requiring that the solution be a continuous function of the parameters as in reference 32, and excluding those roots with positive real parts, it is found that only two roots survive. The properties of these can be conveniently studied using an expansion for small values of $\gamma-1$. The results are qualitatively similar to those given in reference 1. One of the roots leads to a wave speed which differs only slightly from that of a classical acoustic wave as in the approximate solution. For values of κb greater than one, the properties of the other root are also given correctly by the approximation. However, the exact characteristic equation (D.4) indicates that at values of κb somewhat less than 1.0, the second root disappears. The approximate solution does not reproduce this effect, but instead indicates that the wave speed approaches infinity at this point. The approximate result is qualitatively correct in spite of this difference, since the integral term in the exact equation (D.1) corresponds to waves with an infinite wave speed, and it is this term that is simulated by the second root in the approximate solution for small values of the product κb.

Equation (D.5) is an integral equation for the amplitude $G(\theta)$ appearing in the integral term of equation (D.1). No solution has been found. However, if it is assumed that one exists, some of the properties of the solution of equation (C.7) can be deduced from the form of equation (D.1).

If an asymptotic expansion of equation (D.5) for small θ is considered, it can be seen that the expansion is of the form

$$G(\theta)=a_2\theta^2+b_3\theta^3\ln\theta+a_3\theta^3+\ldots \quad \text{(D.6)}$$

Substitution of this into equation (D.1) shows that the third derivative of $H(y)$ will be singular at $y=0$ as anticipated in the approximate considerations of appendix C.

Substitution of equation (D.1) into the boundary condition (C.10) yields

$$\sum_{j=1}^{2} c_jC_j-b\int_0^1\frac{G(\theta)}{\theta}\,d\theta=A \quad \text{(D.7)}$$

This relation together with equation (D.5) must determine the values of the amplitudes C_1 and C_2 if a unique solution exists. Since the function $G(\theta)$ has not been found, exact values of the amplitudes C_1 and C_2 are not available. Although the values of these amplitudes are not known, the functional dependence of the solution for large y can be found from equation (D.1) in cases where the real part of c_1 or c_2 is greater than $-b$. In those cases the unknown integral term will be exponentially small compared to the leading term, which is of the complex exponential type. The wave speed parameter for this functional dependence at large y is known exactly, since it is a root of the characteristic equation (D.4).

The characteristic equation (D.4) is contained in a result given in reference 7. There the equation is more complicated, since it includes the effects of viscosity, thermal heat conduction, a finite velocity of light, and a frequency dependent absorption coefficient. No attempt is made in reference 7 to describe the source of the disturbance or include the effect of a wall. Instead, the development is based on conditions far from any obstacle. Also the investigation of the roots of the characteristic equation is confined to a study of the one root corresponding to a modified-classical wave.

Since an exact solution of the present problem would be obtained if equation (D.5) could be solved, it is of interest to find whether an equation of this type has been discussed in the literature. As it stands, equation (D.5) contains the two unknown constants C_1 and C_2. If a solution to the complete problem exists and is unique, C_1 and C_2 must be determined by equations (D.5) and (D.7). It can be seen in equation (D.5) that $G(1)$ must be zero because of the singularity in the log term. Using this fact, a relation between the given constants A and B can be found which will result in $C_2=0$. A similar relation leading to $C_1=0$ can also be found. When either of these conditions is satisfied, a fundamental solution is obtained containing only a C_1 term or only a C_2 term plus the integral term involving $G(\theta)$.

The integral equation for $G(\theta)$ in either case no longer contains unknown constants. For arbitrary values of A and B, the solution will then be a superposition of the two fundamental solutions. The integral equations for $G(\theta)$ corresponding to the two fundamental solutions can be expressed in terms of the functions $f_1(\theta)$ and $f_2(\theta)$ as follows

$$\left\{\gamma\left(\frac{\theta^2+b^2}{\gamma\theta^2+b^2}\right)-i\kappa b\left[2-\theta\ln\left(\frac{1+\theta}{1-\theta}\right)\right]\right\}f_1(\theta)$$
$$=i\kappa b\left\{\left(\frac{1+\frac{c_1^2}{\gamma}}{b+c_1}\right)\left[1+\gamma b\int_0^1\frac{(1-\theta)\theta}{\gamma\theta^2+b^2}f_1(\theta)d\theta\right]\right.$$
$$\left.\left(\frac{b}{b+c_1\theta}\right)+\int_0^1 f_1(\mu)\left(\frac{\mu}{\mu-\theta}\right)d\mu\right\} \quad \text{(D.8)}$$

$$\left\{\gamma\left(\frac{\theta^2+b^2}{\gamma\theta^2+b^2}\right)-i\kappa b\left[2-\theta\ln\left(\frac{1+\theta}{1-\theta}\right)\right]\right\}f_2(\theta)$$
$$=i\kappa b\left\{\left[1+\int_0^1 f_2(\theta)\theta d\theta\right]\left(\frac{c_2}{b+c_2\theta}\right)\right.$$
$$\left.+\int_0^1 f_2(\mu)\left(\frac{\mu}{\mu-\theta}\right)d\mu\right\} \quad \text{(D.9)}$$

$G(\theta)$ is given in terms of the solutions of these equations, and the specified constants A and B, by the relations

$$G(\theta)=\left(\frac{A-Ba}{1-ad}\right)\frac{\gamma(1-\theta)\theta^2}{(\gamma\theta^2+b^2)}f_1(\theta)$$
$$+\left(\frac{B-Ad}{1-ad}\right)\frac{\gamma(1-\theta)\theta^2}{(\gamma\theta^2+b^2)}f_2(\theta) \quad \text{(D.10)}$$

where

$$a=-\gamma b\int_0^1\frac{(1-\theta)\theta}{(\gamma\theta^2+b^2)}f_2(\theta)d\theta$$
$$+\left(\frac{\gamma c_2}{\gamma+c_2^2}\right)\left(\frac{b+c_2}{b}\right)\left[1+\int_0^1\theta f_2(\theta)d\theta\right] \quad \text{(D.11)}$$

$$d=-\int_0^1\theta f_1(\theta)d\theta+\left(\frac{\gamma+c_1^2}{\gamma c_1}\right)\left(\frac{b}{b+c_1}\right)$$
$$\left[1+\gamma b\int_0^1\frac{(1-\theta)\theta}{(\gamma\theta^2+b^2)}f_1(\theta)d\theta\right] \quad \text{(D.12)}$$

The amplitudes C_1 and C_2 are determined by

$$C_1=\frac{1}{c_1}\left[A+\gamma b\int_0^1\frac{(1-\theta)\theta}{(\gamma\theta^2+b^2)}f_1(\theta)d\theta\right] \quad \text{(D.13)}$$

$$C_2=\left(\frac{\gamma}{\gamma+c_2^2}\right)\left(\frac{b+c_2}{b}\right)\left[B+\int_0^1\theta f_2(\theta)d\theta\right] \quad \text{(D.14)}$$

For values of κb such that c_2 does not exist, the correct result is obtained by setting $c_2=0$. It then follows that $f_2(\theta)=0$ and $C_2=0$.

Solution of the complete problem now depends on solution of the integral equations (D.8) and (D.9) for the unknown functions $f_1(\theta)$ and $f_2(\theta)$. These two independent equations are essentially of the same form. It was pointed out to the author by Harvard Lomax, of Ames Research Center, that a general solution for equations of this type is given in reference 33. However, one of the conditions used in the derivation is not satisfied by equations (D.8) and (D.9), because of the singularity at $\theta=1$ in the log terms. Therefore, it is not known whether the solution is valid in the present case. Alternatively, if it is assumed that $f(\theta)$ is bounded in the interval $0\leq\theta\leq1$, an approximate solution could be obtained by means of a truncated expansion of $f(\theta)$ in Legendre polynomials. The result from this procedure would be of interest for comparison with the results from the exponential approximation of the attenuation factor used in the text, but the matter will not be pursued here.

APPENDIX E

EVALUATION OF INTEGRALS FOR LIMITING CASES IN THE IMPULSIVE-PISTON PROBLEM

In the text it was shown that, for the impulsive problem, all disturbance quantities are zero at slightly positive values of τ. The amplitudes of steps which occur in the velocity, pressure, and temperature were also evaluated. Using similar methods, the variations of the flow quantities at a point far from the wall can be found. The behavior of the solution for very large values of the radiation parameter k can also be found in closed form. Finally, a result for very large τ can be derived.

For the velocity at a point far from the wall, the quantity to be evaluated is

$$\lim_{\xi\to\infty} \frac{u(\tau,\xi)}{U} = \lim_{\xi\to\infty} \text{Re}\, \frac{-i}{\pi} \int_0^\infty (A_1 e^{c_1\nu\xi} + A_2 e^{c_2\nu\xi}) e^{i\nu\tau} \frac{d\nu}{\nu}$$

$$= \lim_{\xi\to\infty} \text{Re}\, \frac{-i}{\pi} \left[\int_0^{\xi^{-1/4}} + \int_{\xi^{-1/4}}^{\xi} + \int_\xi^\infty (A_1 e^{c_1\nu\xi} + A_2 e^{c_2\nu\xi}) e^{i\nu\tau} \frac{d\nu}{\nu} \right] \quad \text{(E.1)}$$

As discussed earlier, the integral near $\nu=0$ is taken on an infinitesimal quarter circle below the origin. We wish to show that the contribution to the integral from values of ν greater than $\xi^{-1/4}$ goes to zero in the limit as ξ goes to infinity. This is so because the real parts of $c_1\nu\xi$ and $c_2\nu\xi$ go to minus infinity in the limit, but the proof requires a knowledge of the behavior of the real parts of c_1 and c_2 as functions of ν for all k.

For the purpose just stated, it can be shown that

$$\left.\begin{aligned} \text{Re}\, c_1\nu\xi &\lessapprox -\frac{\gamma-1}{\gamma+1} k\sqrt{\xi} + \epsilon(\xi) \\ \text{Re}\, c_2\nu\xi &\lessapprox -\frac{1}{2}\frac{\sqrt{\gamma+1}}{k} \xi^{3/4} + \epsilon(\xi) \end{aligned}\right\} \quad \begin{aligned} &\text{for } \xi^{-1/4} \lessapprox \nu \lessapprox \xi \\ &\text{and } k\neq 0, k \text{ finite} \end{aligned}$$

where $\epsilon(\xi)$ is a quantity which goes to zero in the limit as ξ goes to infinity. It follows that

$$\left.\begin{aligned} |e^{c_1\nu\xi}| &\lessapprox \exp\left\{ -\left[\frac{\gamma-1}{\gamma+1} k\sqrt{\xi} - \epsilon(\xi)\right]\right\} \\ |e^{c_2\nu\xi}| &\lessapprox \exp\left\{ -\left[\frac{1}{2}\frac{\sqrt{\gamma+1}}{k} \xi^{3/4} - \epsilon(\xi)\right]\right\} \end{aligned}\right\} \quad \begin{aligned} &\text{for } \xi^{-1/4} \lessapprox \nu \lessapprox \xi \\ &\text{and } k\neq 0, k \text{ finite} \end{aligned}$$

Since A_1 and A_2 are algebraic functions of ν, the contribution to the integral from integration in the range $\nu=\xi^{-1/4}$ to $\nu=\xi$ will go to zero exponentially in the limit as ξ goes to infinity provided that k is finite and nonzero.

The contribution to the integral from values of ν between ξ and ∞ can be shown to be zero in the limit by expansion of the integrand for large ν. Then equation (E.1) becomes

$$\lim_{\xi\to\infty} \frac{u(\tau,\xi)}{U} = \lim_{\xi\to\infty} \text{Re}\, \frac{-i}{\pi} \int_0^{\xi^{-1/4}} (A_1 e^{c_1\nu\xi} + A_2 e^{c_2\nu\xi}) e^{i\nu\tau} \frac{d\nu}{\nu} \quad \text{(E.2)}$$

This integral can be evaluated by expansion of the integrand for small ν. Expansion of equations (82), (83), (88) and (89) for small ν yields

$$c_1 = -i - \frac{\gamma-1}{\gamma+1} k\nu + 0(\nu^2)$$

$$c_2 = -\left(\frac{1+i}{2}\right)\sqrt{\frac{\gamma+1}{k\nu}} + 0(\sqrt{\nu})$$

$$A_1 = 1 + 0(\sqrt{\nu})$$

$$A_2 = 0(\sqrt{\nu})$$

Substitution of these into equation (E.2) gives

$$\lim_{\xi\to\infty} \frac{u(\tau,\xi)}{U} = \lim_{\xi\to\infty} \text{Re}\, \frac{-i}{\pi} \int_0^{\xi^{-1/4}} \left\{ e^{-i\nu\xi} \exp\left[-\left(\frac{\gamma-1}{\gamma+1}\right) k\nu^2\xi\right] + 0(\sqrt{\nu}) \right\} e^{i\nu\tau} \frac{d\nu}{\nu}$$

The integral of the $0(\sqrt{\nu})-$ term is less than $0(\xi^{-1/8})$ and hence goes to zero in the limit, leaving

$$\lim_{\xi\to\infty}\frac{u(\tau,\xi)}{U}=\lim_{\xi\to\infty}\mathrm{Re}\,\frac{-i}{\pi}\int_0^{\xi^{-1/4}}\exp\left[-\left(\frac{\gamma-1}{\gamma+1}\right)k\xi\nu^2\right]e^{i\nu(\tau-\xi)}\frac{d\nu}{\nu}$$

By reversing the steps leading to this expression, the full interval of integration can be restored and the result written as

$$\lim_{\xi\to\infty}\frac{u(\tau,\xi)}{U}=\lim_{\xi\to\infty}\left\{\frac{-i}{2\pi}\int_{-\infty}^{\infty}\left\{\exp\left[-\left(\frac{\gamma-1}{\gamma+1}\right)k\xi\nu^2\right]-1\right\}e^{i\nu(\tau-\xi)}\frac{d\nu}{\nu}-\frac{i}{2\pi}\int_{-\infty}^{\infty}e^{i\nu(\tau-\xi)}\frac{d\nu}{\nu}\right\}\quad\text{(E.3)}$$

The second term can be evaluated by contour integration and corresponds to a unit step at $\tau=\xi$. In this process the path of integration is taken to pass below the origin as discussed earlier. The first integral in equation (E.3) can be found in reference 34. The final result for the velocity at large distance from the wall is

$$\frac{u(\tau,\xi)}{U}=\frac{1}{2}\left(1+\mathrm{erf}\left\{\frac{\tau-\xi}{2\left[\left(\frac{\gamma-1}{\gamma+1}\right)k\xi\right]^{1/2}}\right\}\right)\quad(\xi\to\infty)\quad\text{(E.4)}$$

Equation (E.4) indicates a smooth transition from a value of zero at $\tau=0$ to a value of 1.0 when τ becomes large compared to ξ. The transition is essentially completed within a range of values of τ from $\tau=\xi-\sqrt{\left(\frac{\gamma-1}{\gamma+1}\right)k\xi}$ to $\tau=\xi+\sqrt{\left(\frac{\gamma-1}{\gamma+1}\right)k\xi}$, which is a narrow region compared to the large distance from the wall under consideration.

In the derivation of equation (E.4), the integral of the A_2 term was found to be zero. It is somewhat surprising that an integral involving only the A_1 term yields a nonzero value of the disturbance at $\tau-\xi<0$. The results from the solution of the oscillating piston problem indicate that the maximum possible velocity of the modified classical waves is the isentropic speed of sound a_0. In the present problem the A_1 term represents the contribution from the modified-classical wave system, and in terms of the dimensionless coordinates τ, ξ it follows that the maximum velocity of such waves is 1.0. The region $\tau-\xi<0$ is a part of the τ, ξ plane which cannot be reached by waves initiated at $\tau=0$, $\xi=0$ and traveling at a maximum speed of 1.0. The A_2 term, on the other hand, represents the contribution from the radiation-induced wave system. The oscillating-piston solution indicates that the maximum speed of these waves is the velocity of light (taken to be infinite). Hence the region $\tau-\xi<0$ can be reached by waves associated with the A_2 term, but not by those associated with the A_1 term.

The foregoing considerations tend to cast doubt on the choice of the real axis in the complex ν plane as the path of integration for inversion of the Fourier transform of $u(\tau, \xi)$. Nevertheless, this choice was shown to correspond to the boundary conditions and initial conditions for the problem under consideration (in the discussion following equation (67)). In that discussion it was noted that a branch cut associated with the dependence of c_1 and c_2 on complex values of ν could be disregarded as far as the integrand is concerned. This is true because the quantity $A_1 \exp(c_1\nu\xi)+A_2 \exp(c_2\nu\xi)$ is continuous across the branch cut in question, which is the one in the lower half plane. However, $A_1 \exp(c_1\nu\xi)$ alone is not continuous across this cut, but instead interchanges roles with the other term $A_2 \exp(c_2\nu\xi)$ in passing from one side of the cut to the other. As a result, if we wish to compute the contribution from the modified classical wave system alone, the path of integration must be altered to include an integration around the branch cut in order to insure that the disturbance associated with this wave system is zero for negative time. An integration around the branch cut would also be required to evaluate the contribution from the radiation-induced wave system alone. This additional contribution from the A_2 term would not be zero, even though the integral of the A_2 term along the real axis is zero. Instead, the additional contribution from the A_2 term is just equal and opposite to the additional contribution from integration of the A_1 term around the branch cut. This last fact explains why, in evaluating the total disturbance, the integration around the branch cut can be dispensed with. But it also means that part of the total disturbance arises from the A_2 term, even though the integral of the A_2 term along the real axis is zero. Then it is the radiation-induced wave system which is responsible for the disturbance at $\tau-\xi<0$, a region which cannot be reached by the modified classical waves.

The corresponding expressions for pressure and temperature far from the wall are similar to equation (E.4); namely,

$$\frac{p'(\tau,\xi)}{\rho_0 a_0 U}=\frac{1}{2}\left[1+\operatorname{erf}\left\{\frac{\tau-\xi}{2\left[\left(\frac{\gamma-1}{\gamma+1}\right)k\xi\right]^{1/2}}\right\}\right] \quad (\xi\rightarrow\infty) \quad \text{(E.5)}$$

$$\frac{RT'(\tau,\xi)}{a_0 U}=\frac{1}{2}\,\frac{\gamma-1}{\gamma}\left[1+\operatorname{erf}\left\{\frac{\tau-\xi}{2\left[\left(\frac{\gamma-1}{\gamma+1}\right)k\xi\right]^{1/2}}\right\}\right]+0\left(\frac{k\tau}{\xi^2}\right) \quad (\xi\rightarrow\infty) \quad \text{(E.6)}$$

The term of order $(k\tau/\xi^2)$ in equation (E.6) comes from the A_2 part of equation (80). For very large τ of order ξ^2/k, this term cancels the other part of equation (E.6), and the perturbation temperature returns to zero. The velocity and pressure do not change in this process. Because of the homogeneity of the gas and the presence of the radiative heat transfer process, the gas must reach a uniform state at very large time. In this state the velocity and temperature of the gas must be the same as the velocity and temperature of the wall. Since the wall temperature is held fixed while the wall is moved impulsively, the final perturbation temperature is zero. It is interesting to note that the final perturbation pressure is the same as that which would occur in the absence of radiation, even though the final temperature is not.

Another evaluation can be made in closed form as a check on the numerical calculations which will be discussed later. At very large values of k (high gas temperature) the radiative heat transfer will tend to hold the temperature constant. Under this condition, one would expect the disturbance to propagate at the isothermal speed $a_0/\sqrt{\gamma}$. In terms of dimensionless coordinates, the velocity transition would then be expected to occur near the line $\tau=\sqrt{\gamma}\xi$. It can be shown that this is so for intermediate values of ξ, but not at very small or very large ξ. For this purpose equation (78) can be written as

$$\lim_{k\rightarrow\infty}\frac{u(\tau,\xi)}{U}=\lim_{k\rightarrow\infty}\operatorname{Re}\frac{-i}{\pi}\left[\int_0^{1/k}+\int_{1/k}^{k}+\int_k^{\infty}(A_1e^{c_1\nu\xi}+A_2e^{c_2\nu\xi})e^{i\nu\tau}\frac{d\nu}{\nu}\right]$$
$$=I_1+I_2+I_3 \quad \text{(E.7)}$$

The first integral, which includes integration along an infinitesimal quarter circle about the origin, can be evaluated in closed form for large k. The second integral can also be evaluated by means of an expansion valid for large k. The remaining integral can be shown to be zero in the limit as k goes to infinity with ξ fixed.

In disposing of the first and last integrals, a knowledge of the properties of the quantities c_1 and c_2 as functions of ν, k, and γ is needed. Except for an infinitesimal deviation below the origin, only real values of ν need be considered. Values of k from zero to infinity, and values of γ between 1.0 and 2.0 are of interest. From equation (81), the following properties can be deduced for the foregoing ranges of ν, k, γ:

1. The complex quantity c_1 has no singularities (when the value at $\nu=0$ is properly defined).
2. The real part of c_1 has zeros at $\nu=0$ and at $\nu=\infty$.
3. The imaginary part of c_1 has no zeros.
4. The quantity c_2 has a singularity at $\nu=0$.
5. Both real and imaginary parts of c_2 have zeros at $\nu=\infty$.
6. The real and imaginary parts of c_1 and c_2 are negative or zero.

By factoring out the singularities and expanding about the singularities and zeros, it can be established that

$$-i\left[1-a_1(\gamma-1)\frac{ik\nu}{1+\nu^2-ik\nu}\right]<c_1<-i\left[1-b_1(\gamma-1)\frac{ik\nu}{1+\nu^2-ik\nu}\right] \quad \text{(E.8)}$$

and

$$-\frac{a_2}{\sqrt{\nu(\nu-ik)}}<c_2<-\frac{b_2}{\sqrt{\nu(\nu-ik)}} \quad \text{(E.9)}$$

where a_1, b_1, a_2, b_2 are positive finite nonzero numbers which can be chosen to be independent of

ν, k, and γ. To establish that a and b can be independent of γ and k for values of k approaching zero or infinity, use can be made of a power series expansion of equation (81) in powers of $(\gamma-1)$. The resulting power series converges for the values of ν, k, and γ under consideration.

The content of equations (E.8) and (E.9) can be expressed by the relations

$$c_1 \approx -i\left[1-b_1(\gamma-1)\frac{ik\nu}{(1+\nu^2-ik\nu)}\right] \quad \text{(E.10)}$$

$$c_2 \approx -\frac{b_2}{\sqrt{\nu(\nu-ik)}} \quad \text{(E.11)}$$

where b_1 and b_2 are independent of ν, k, and γ. Equations (E.10) and (E.11) can then be used as upper or lower bounds with appropriate values of b_1 and b_2. These expressions for c_1 and c_2 are of the same form as those resulting from a truncated expansion for small $(\gamma-1)$ used in reference 1.

The same procedure can be applied to the expressions for A_1 and A_2 using equations (E.10) and (E.11) to obtain the relations

$$A_1 \approx 1-b_3(\gamma-1)\frac{(\nu-i)\sqrt{\nu}(\sqrt{\nu-ik}-\sqrt{\nu})}{(1+\nu^2-ik\nu)^2} \quad \text{(E.12)}$$

$$A_2 \approx b_3(\gamma-1)\frac{(\nu-i)\sqrt{\nu}(\sqrt{\nu-ik}-\sqrt{\nu})}{(1+\nu^2-ik\nu)^2} \quad \text{(E.13)}$$

These expressions provide upper bounds for the quantity $1-A_1=A_2$ with b_3 independent of ν, k, and γ.

To evaluate the first integral in equation (E.7), equations (E.10)–(E.13) can be expanded for small ν. The results are

$$c_1 \approx -i-\frac{b_1(\gamma-1)k\nu}{1-ik\nu}+0(\nu^2) \quad \text{(E.14)}$$

$$c_2 \approx -(b_2/\sqrt{-ik\nu})[1+0(\nu/k)] \quad \text{(E.15)}$$

$$A_2=1-A_1 \approx \frac{b_3(\gamma-1)\sqrt{ik\nu}}{(1-ik\nu)^2}\left[1+0\left(\sqrt{\frac{\nu}{k}}\right)+0(\nu^2)\right] \quad \text{(E.16)}$$

Substitution of these into equation (E.7), replacing $k\nu$ with r and expanding for small values of τ/k and ξ/k yields

$$I_1=\lim_{k\to\infty} \mathrm{Re}\,\frac{-i}{\pi}\int_0^1\left\{\left[1+ir\left(\frac{\tau}{k}-\frac{\xi}{k}\right)-b_1(\gamma-1)\frac{\xi}{k}\frac{r^2}{1-ir}\right]\left[1-b_3\frac{(\gamma-1)\sqrt{ir}}{(1-ir)^2}+0\left(\frac{r}{k}\right)\right]+b_3(\gamma-1)\frac{\sqrt{ir}}{(1-ir)^2}\left[1+ir\frac{\tau}{k}-b_2\frac{\xi}{k}\sqrt{ir}+0\left(\frac{\sqrt{r}}{k}\right)\right]+0\left[\left(\frac{\tau}{k}\right)^2,\left(\frac{\xi}{k}\right)^2\right]\right\}\frac{dr}{r}$$

Evaluation of this integral, including the infinitesimal quarter circle about the origin, leads to the result

$$I_1=\frac{1}{2}+\lim_{k\to\infty} 0\left(\frac{\tau}{k},\frac{\xi}{k}\right) \quad \text{(E.17)}$$

Evaluation of the third integral in equation (E.7) can be accomplished by expansion of equations (E.10)-(E.13) for large ν as follows:

$$c_1 \approx -i-b_1(\gamma-1)\frac{k}{\nu-ik}\left[1+0\left(\frac{1}{\nu^2}\right)\right] \quad \text{(E.18)}$$

$$c_2 \approx -b_2/\sqrt{\nu(\nu-ik)} \quad \text{(E.19)}$$

$$A_2=1-A_1 \approx 0\left(\frac{1}{\nu^2}\right) \quad \text{(E.20)}$$

$$I_3=\lim_{k\to\infty} \mathrm{Re}\,\frac{-i}{\pi}\int_k^\infty\left\{e^{i\nu(\tau-\xi)}\exp\left[-b_1(\gamma-1)\frac{k\nu\xi}{\nu-ik}\right]+0\left(\frac{1}{\nu^2}\right)\right\}\frac{d\nu}{\nu} \quad \text{(E.21)}$$

The integral of the $0\left(\frac{1}{\nu^2}\right)\frac{1}{\nu}$ term goes to zero in the limit. The remainder can be evaluated by separating the argument of the second exponential factor into real and imaginary parts. After the substitution $s=\nu/k$ is made, the result is

$$I_3=\lim_{k\to\infty} \mathrm{Re}\,\frac{-i}{\pi}\int_1^\infty e^{is(k\tau-k\xi)}\,e^{-b_1(\gamma-1)k\xi/(1+s^{-2})}\,e^{-ib_1(\gamma-1)k\xi s/(1+s^2)}\frac{ds}{s}$$

Since b_1 is of order one, this becomes

$$I_3=\lim_{k\to\infty} 0\,[e^{-(\gamma-1)k\xi}] \quad \text{(E.22)}$$

For the remaining part of equation (E.7), the exact expressions for c and A given in equations

(82), (83), (88) and (89) must be used. However, an expansion for large k with ν fixed can be made as follows:

$$c_1=-i\sqrt{\gamma}-\frac{\gamma-1}{2\sqrt{\gamma}}\left(\frac{\frac{\gamma+1}{2\gamma}+\nu^2}{k\nu}\right)+0\left(\frac{1}{k^2\nu},\frac{1}{k^2},\frac{\nu^2}{k^2}\right) \quad \text{(E.23)}$$

$$c_2=0\left(\frac{1}{\sqrt{k\nu}}\right) \quad \text{(E.24)}$$

$$A_2=1-A_2=0\left[\left(\frac{1}{k\nu}\right)^{3/2},\frac{\sqrt{\nu}}{k^{3/2}}\right] \quad \text{(E.25)}$$

$$I_2=\lim_{k\to\infty}\text{Re}\,\frac{-i}{\pi}\int_{k^{-1}}^{k}[e^{c_1\nu\xi}+A_2(e^{c_2\nu\xi}-e^{c_1\nu\xi})]e^{i\nu\tau}\,\frac{d\nu}{\nu} \quad \text{(E.26)}$$

By breaking the integral into two parts, $1/k$ to 1.0 plus 1.0 to k, and expanding the exponentials in power series, the term containing A_2 is found to be at most of order ξ/k. Similarly, the higher order term in equation (E.23) can be shown to lead to terms which are at most of order $\sqrt{1/(\gamma-1)^3k\xi}$. Then equation (E.26) becomes

$$I_2=\lim_{k\to\infty}\left\{\text{Re}\,\frac{-i}{\pi}\int_{k^{-1}}^{k}e^{i\nu(\tau-\sqrt{\gamma}\xi)}\exp\left[-\left(\frac{\gamma-1}{2\gamma}\right)\xi\,\frac{\nu^2}{k}\right]\frac{d\nu}{\nu}+0\left[\frac{\xi}{k},\sqrt{1/(\gamma-1)^3k\xi}\right]\right\} \quad \text{(E.27)}$$

By reversing the procedure in which the I_1 and I_3 integrals were removed, the full interval of integration can be restored and I_2 written as

$$I_2=\lim_{k\to\infty}\left(\frac{-i}{2\pi}\int_{-\infty}^{\infty}e^{i\nu(\tau-\sqrt{\gamma}\xi)}\exp\left[-\left(\frac{\gamma-1}{2\sqrt{\gamma}}\right)\xi\,\frac{\nu^2}{k}\right]\frac{d\nu}{\nu}+0\left\{\frac{\xi}{k},[(\gamma-1)^3k\xi]^{-1/2}\right\}\right) \quad \text{(E.28)}$$

This integral is of the same form as that evaluated in equation (E.3).

Finally, the velocity at large k is found to be

$$\lim_{k\to\infty}\frac{u(\tau,\xi)}{U}=\lim_{k\to\infty}\left[\frac{1}{2}\left(1+\text{erf}\left\{\frac{\tau-\sqrt{\gamma}\xi}{2\left[\left(\frac{\gamma-1}{2\sqrt{\gamma}}\right)\frac{\xi}{k}\right]^{1/2}}\right\}\right)+0\left\{\frac{\xi}{k},[(\gamma-1)^3k\xi]^{-1/2}\right\}\right] \quad \text{(E.29)}$$

The corresponding results for pressure and temperature are

$$\lim_{k\to\infty}\frac{p'(\tau,\xi)}{\rho_0a_0U}=\lim_{k\to\infty}\left[\frac{1}{2\sqrt{\gamma}}\left(1+\text{erf}\left\{\frac{\tau-\sqrt{\gamma}\xi}{2\left[\left(\frac{\gamma-1}{2\sqrt{\gamma}}\right)\frac{\xi}{k}\right]^{1/2}}\right\}\right)+0\left\{\frac{\tau}{k},\frac{\xi}{k},[(\gamma-1)^3k\xi]^{-1/2}\right\}\right] \quad \text{(E.30)}$$

and

$$\lim_{k\to\infty}\frac{RT'(\tau,\xi)}{a_0U}=\lim_{k\to\infty}\left(0\left\{\frac{\tau}{k},\frac{\xi}{k},[(\gamma-1)^3k\xi]^{-1/2}\right\}\right) \quad \text{(E.31)}$$

Equations (E.29)–(E.31) represent the isothermal disturbance propagating at the isothermal signal velocity as was anticipated. The error function transition occurs in a narrow region with width of order ξ/k, which goes to zero in the limit as k goes to infinity with ξ fixed at a finite, nonzero value. In the derivation of these relations, it was necessary to neglect terms of order τ/k, ξ/k and $\sqrt{1/(\gamma-1)^3k\xi}$. In fact, the results given here do not apply at values of ξ of order k^{-1}, where there is a boundary layer. Also at large values of τ and ξ of order k, different results are obtained.

Expressions valid for small ξ were given in equations (94)–(96). Those results are also valid for the present case of large k provided that ξ is small compared to k^{-1}. The velocity, temperature and pressure jumps occurring along the line $\tau=\xi$, evaluated in equations (97)–(99), are still correctly given in the case of large k. These discontinuities are within the boundary layer which develops for large k, since they decay exponentially in a distance of order k^{-1} from the wall. Also the results for large ξ given in equations (E.4)–(E.6) are valid in the limit as k goes to infinity, with ξ going to infinity faster.

The values of velocity, pressure, and temperature for very large time can be found by taking the limit as τ goes to infinity. The factor $e^{i\nu\tau}$ then oscillates rapidly, and the contribution to each integral vanishes for all values of ν except those near singularities in the remaining factors of each ntegrand. The only such singularities are at $\nu=0$. Expansion of equations (82), (83), (88), and (89) about $\nu=0$ and integration of equations (78)–(80) yields

$$\lim_{\tau\to\infty}\frac{u(\tau,\xi)}{U}=1.0 \tag{E.32}$$

$$\lim_{\tau\to\infty}\frac{p'(\tau,\xi)}{\rho_0 a_0 U}=1.0 \tag{E.33}$$

$$\lim_{\tau\to\infty}\frac{RT'(\tau,\xi)}{a_0 U}=0 \tag{E.34}$$

APPENDIX F

APPROXIMATE CLOSED-FORM EVALUATION OF THE VELOCITY RESPONSE TO AN IMPULSIVE MOTION OF A PISTON

In reference 1, the response to sinusoidal motion of the wall was discussed with the aid of a truncated expansion for small values of $(\gamma-1)$. The same procedure is useful for obtaining further information on the impulsive-motion problem. Equations (88) and (89) can be interpreted as an expansion for small values of $(\gamma-1)$ in addition to the previously given interpretations of small ν, large ν, and large k. The expansion converges for values of γ between 1.0 and 2.0. If terms of order $(\gamma-1)^2$ are neglected, equations (88) and (89) become

$$c_1=-i\left[1-(\gamma-1)\frac{ik\nu}{\nu^2-ik\nu+1}+0(\gamma-1)^2\right]^{\frac{1}{2}}$$

$$c_2=-\sqrt{\frac{(\gamma+1)/2}{\nu(\nu-ik)}}\left[1+(\gamma-1)\frac{ik\nu}{\nu^2-ik\nu+1}+0(\gamma-1)^2\right]^{\frac{1}{2}}$$

Further expansion of the square root factors yields

$$c_1=-i\left[1-\left(\frac{\gamma-1}{2}\right)\frac{ik\nu}{\nu^2-ik\nu+1}+0(\gamma-1)^2\right]$$

$$c_2=-\sqrt{\frac{(\gamma+1)/2}{\nu(\nu-ik)}}\left[1+\left(\frac{\gamma-1}{2}\right)\frac{ik\nu}{\nu^2-ik\nu+1}+0(\gamma-1)^2\right]$$

These expressions for c_1 and c_2 are identical to equations (68) and (69) of reference 1, except for the use here of the variables k and ν in place of K and β. The equivalence can be established with the aid of equations (39), (40), and (76) herein, and equation (66) of reference 1.

Since $\gamma-1=(\sqrt{\gamma}+1)(\sqrt{\gamma}-1)=2(\sqrt{\gamma}-1)+0(\sqrt{\gamma}-1)^2$, the expressions for c_1 and c_2 can be written alternatively as

$$c_1=-i\left[1-(\sqrt{\gamma}-1)\frac{ik\nu}{\nu^2-ik\nu+1}+0(\sqrt{\gamma}-1)^2\right] \quad \text{(F.1)}$$

$$c_2=-\sqrt{\frac{(\gamma+1)/2}{\nu(\nu-ik)}}\left[1+(\sqrt{\gamma}-1)\frac{ik\nu}{\nu^2-ik\nu+1}+0(\sqrt{\gamma}-1)^2\right] \quad \text{(F.2)}$$

The last forms are chosen instead of the previous ones because the resulting truncated expansion is exact for $k\rightarrow\infty$ as well as for $k\rightarrow0$, $\nu\rightarrow\infty$, and $\nu\rightarrow0$. Using equations (F.1) and (F.2) in a corresponding expansion of equations (82) and (83) yields

$$A_2-1-A_1=\frac{2(\sqrt{\gamma}-1)}{\gamma}\frac{(\nu-i)\sqrt{\nu}(\sqrt{\nu-ik}-\sqrt{\nu})}{(\nu^2-ik\nu+1)^2}+0(\sqrt{\gamma}-1)^2 \quad \text{(F.3)}$$

The factor γ in the denominator could be deleted, but this would increase the value of the error in the truncated expansion by a rather large factor for a value of $\gamma=1.40$. The coefficients of the higher order terms in equations (F.1)–(F.3) have maximum values of approximately 1.0 and are zero for limiting values of k and ν.

There is another precedent for an approximation of the foregoing type in addition to that afforded in reference 1. The isothermal signal speed is $a_0/\sqrt{\gamma}$ where a_0 is the isentropic speed of sound. Thus an expansion for small values of $(\sqrt{\gamma}-1)$ can be interpreted as an expansion for a small fractional difference between the isothermal and isentropic speeds. In reference 35 a similar approximation is introduced for the case of a chemically relaxing gas in the absence of radiation. There the approximation is based on the smallness of the fractional difference between the frozen and equilibrium speeds of sound. Comparison of the results of reference 35 with the exact results of references 10 and 31 shows the approximation to be accurate to within a few percent for the cases considered.

Estimates of the error introduced by this approximation in the present problem can be obtained by repeating the closed-form evaluations for limiting cases previously made. For the jump conditions at $\tau=\xi$, the following results are obtained for comparison with equations (97)–(99):

$$\lim_{\epsilon\rightarrow0}\left[\frac{u(\xi+\epsilon,\xi)}{U}-\frac{u(\xi-\epsilon,\xi)}{U}\right]=e^{-(\sqrt{\gamma}-1)k\xi}+0(\sqrt{\gamma}-1)^2 \quad \text{(F.4)}$$

$$\lim_{\epsilon\to 0}\left[\frac{p'(\xi+\epsilon,\xi)}{\rho_0 a_0 U}-\frac{p'(\xi-\epsilon,\xi)}{\rho_0 a_0 U}\right]=e^{-(\sqrt{\gamma}-1)k\xi}+0(\sqrt{\gamma}-1)^2 \quad \text{(F.5)}$$

$$\lim_{\epsilon\to 0}\left[\frac{RT'(\xi+\epsilon,\xi)}{a_0 U}-\frac{RT'(\xi-\epsilon,\xi)}{a_0 U}\right]=\left(\frac{\gamma-1}{\gamma}\right)e^{-(\sqrt{\gamma}-1)k\xi}+0(\sqrt{\gamma}-1)^2 \quad \text{(F.6)}$$

The comparison shows that equations (F.4)–(F.6) are exact at $\xi=0$, but the decay factor in the exact equations is $\exp -\left(\frac{\sqrt{\gamma}+1}{2}\right)(\sqrt{\gamma}-1)k\xi$ rather than $\exp -(\sqrt{\gamma}-1)k\xi$ as given by the approximation. For $\gamma=7/5$, the factor $(\sqrt{\gamma}+1)/2$ is equal to 1.092. The discrepancy does not lead to a nonuniformity in the approximation even at large values of $k\xi$, since the discontinuities die out at large $k\xi$. For $\gamma=7/5$ the largest error occurs at $k\xi=5.46$ where the approximate decay factor is 0.368 compared to the exact value of 0.403. The maximum error is then 0.035, which is 3.5 percent of the total velocity transition in the disturbance (equal to the dimensionless wall velocity which is 1.0).

Using the approximate equations (F.1)–(F.3), the disturbance at large ξ is found to be

$$\lim_{\xi\to\infty}\frac{u(\tau,\xi)}{U}=\frac{1}{2}\left\{1+\operatorname{erf}\left[\frac{\tau-\xi}{2\sqrt{(\sqrt{\gamma}-1)k\xi}}\right]\right\}+0(\sqrt{\gamma}-1)^2 \quad \text{(F.7)}$$

$$\lim_{\xi\to\infty}\frac{p'(\tau,\xi)}{\rho_0 a_0 U}=\frac{1}{2}\left\{1+\operatorname{erf}\left[\frac{\tau-\xi}{2\sqrt{(\sqrt{\gamma}-1)k\xi}}\right]\right\}+0(\sqrt{\gamma}-1)^2 \quad \text{(F.8)}$$

$$\lim_{\xi\to\infty}\frac{RT'(\tau,\xi)}{a_0 U}=\frac{1}{2}\,\frac{\gamma-1}{\gamma}\left\{1+\operatorname{erf}\left[\frac{\tau-\xi}{2\sqrt{(\sqrt{\gamma}-1)k\xi}}\right]\right\}+0(\sqrt{\gamma}-1)^2 \quad \text{(F.9)}$$

Comparison with the exact equations (E.4)–(E.6) shows that these results from the truncated expansion are the same except that the factor $\sqrt{(\sqrt{\gamma}-1)k\xi}$ in the denominator of the argument of the error function is replaced by $\sqrt{\frac{\sqrt{\gamma}+1}{\gamma+1}(\sqrt{\gamma}-1)k\xi}$ in the exact equations. For $\gamma=7/5$, the factor $\frac{\sqrt{\gamma}+1}{\gamma+1}$ is equal to 0.910, representing an error of about 10 percent. However, again the maximum resulting error in the perturbation quantities is a considerably smaller fraction (2 percent) of the total change in the transition. Since approximate expressions for the wave speeds were used to derive equations (F.7)–(F.9), cumulative errors might have been expected to cause large discrepancies at $\xi\to\infty$. However, this does not occur because the components with the largest errors in wave speed are sufficiently damped that their amplitudes become negligible before the error in position becomes appreciable.

The approximate results for the limit as k goes to infinity with τ and ξ fixed are

$$\lim_{k\to\infty}\frac{u(\tau,\xi)}{U}=\lim_{k\to\infty}\frac{1}{2}\left\{1+\operatorname{erf}\left[\frac{\tau-\sqrt{\gamma}\xi}{2\sqrt{(\sqrt{\gamma}-1)\xi/k}}\right]\right\}+0(\sqrt{\gamma}-1)^2 \quad \text{(F.10)}$$

$$\lim_{k\to\infty}\frac{p'(\tau,\xi)}{U}=\lim_{k\to\infty}\frac{1}{2\sqrt{\gamma}}\left\{1+\operatorname{erf}\left[\frac{\tau-\sqrt{\gamma}\xi}{2\sqrt{(\sqrt{\gamma}-1)\xi/k}}\right]\right\}+0(\sqrt{\gamma}-1)^2 \quad \text{(F.11)}$$

$$\lim_{k\to\infty}\frac{RT'(\tau,\xi)}{a_0 U}=0(\sqrt{\gamma}-1)^2 \quad \text{(F.12)}$$

Comparison with the exact equations (E.29)–(E.31) shows these results to be correct except that the factor $\sqrt{(\sqrt{\gamma}-1)\xi/k}$ in the denominator of the error function is replaced by

$$\sqrt{\left(\frac{\sqrt{\gamma}+1}{2\sqrt{\gamma}}\right)(\sqrt{\gamma}-1)\frac{\xi}{k}}$$

in the exact relations. With $\gamma=7/5$, the factor $(\sqrt{\gamma}+1)/2\sqrt{\gamma}$ is equal to 0.923 representing an error of 8 percent in position, but only 2 percent in value at a given position.

The results for the limit as τ goes to infinity compared with the exact equations (E.32) and (E.33) show the approximation to be exact in this limit. The approximate counterpart of equation (E.34) is correct to $0(\sqrt{\gamma}-1)$.

The only remaining exact evaluations available for comparison are those for the limit as τ goes to zero given in equations (94)–(96). There it is seen that the perturbation velocity, temperature, and pressure are zero at $\tau=0$, except at the wall, where there are discontinuities. The truncated expansion for small values of $(\sqrt{\gamma}-1)$ yields small but nonzero values of the perturbation quantities at $\tau\leqq 0$. This slight inconvenience can be removed without affecting the results for the other limiting cases as follows: In the derivation of the result for small τ in the exact case, use was made of the fact that the disturbance is zero at negative τ. The factor $e^{i\nu\tau}$ was replaced by $e^{i\nu\tau}-e^{-i\nu|\tau|}$, the second term leading to zero contribution because it corresponds to an evaluation of the original integral at negative values of τ, where the disturbance is zero. This alteration can also be made in the exact equations (78)–(80), and the results will be unchanged for all values of τ and ξ. In other words, equations (78)–(80) can be replaced with the relations

$$\frac{u(\tau,\xi)}{U}=-\frac{i}{2\pi}\int_{-\infty}^{\infty}(A_1e^{c_1\nu\xi}+A_2e^{c_2\nu\xi})(e^{i\nu\tau}-e^{-i\nu|\tau|})\frac{d\nu}{\nu} \tag{F.13}$$

$$\frac{p'(\tau,\xi)}{\rho_0a_0U}=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{A_1}{c_1}e^{c_1\nu\xi}+\frac{A_2}{c_2}e^{c_2\nu\xi}\right)(e^{i\nu\tau}-e^{i\nu|\tau|})\frac{d\nu}{\nu} \tag{F.14}$$

$$\frac{RT'(\tau,\xi)}{a_0U}=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\left(1+\frac{c_1^2}{\gamma}\right)\frac{A_1}{c_1}e^{c_1\nu\xi}+\left(1+\frac{c_2^2}{\gamma}\right)\frac{A_2}{c_2}e^{c_2\nu\xi}\right](e^{i\nu\tau}-e^{-i\nu|\tau|})\frac{d\nu}{\nu} \tag{F.15}$$

When the truncated expansion for small $(\sqrt{\gamma}-1)$ is used in these equations, the resulting perturbation quantities are zero at $\tau<0$, as they should be. It can be shown that the evaluations of the other limiting cases using the truncated expansion are not affected by this change. Hence the previous comparisons between approximate and exact results remain valid when equations (F.13)–(F.15) are used in place of (78)–(80).

To obtain a qualitative view of the over-all flow field, either further approximation or machine computation is necessary. Both methods will be used. Only the velocity field will be considered in this study. For the machine computations one might expect that the integrals could be evaluated without resorting to the expansion for small values of $(\sqrt{\gamma}-1)$. However, this is not feasible, if a machine program valid for all values of the parameters and variables is desired. For such computations, considerable knowledge of the properties of the integrand are needed for a proper design of the integration procedure. Also the machine computing time required is not negligible. Since the truncated expansion for small values of $(\sqrt{\gamma}-1)$ yields qualitatively correct results, utilizes a simplified integrand, and decreases the required machine computing time by about a factor of ten, it will be used here.

Using equations (F.1)–(F.3), equation (F.13) can be written as

$$\frac{u(\tau,\xi)}{U}=\frac{u_c(\tau,\xi)}{U}+\frac{u_r(\tau,\xi)}{U}+0(\sqrt{\gamma}-1)^2 \tag{F.16}$$

where

$$\frac{u_c(\tau,\xi)}{U}=-\frac{i}{2\pi}\int_{-\infty}^{\infty}(e^{i\nu\tau}-e^{-i\nu|\tau|})e^{-i\nu\xi}e^{-(\sqrt{\gamma}-1)k\xi\nu^2/(1+\nu^2-ik\nu)}\frac{d\nu}{\nu} \tag{F.17}$$

$$\frac{u_r(\tau,\xi)}{U}=-\frac{2(\sqrt{\gamma}-1)}{\gamma}\frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{(e^{i\nu\tau}-e^{-i\nu|\tau|})(\sqrt{1-ik/\nu}-1)(\nu-i)(e^{c_2\nu\xi}-e^{c_1\nu\xi})}{(1+\nu^2-ik\nu)^2}d\nu \tag{F.18}$$

An integral of the form of equation (F.17) cannot be evaluated directly by machine computation because of the singularity at $\nu=0$. Also the infinite interval of integration cannot be treated by machine calculation. Therefore it is necessary to subtract terms from the integrand which match it at $\nu=0$ and at large ν. Such terms should at the same time be simple enough that they can be integrated in closed form. If care is taken in the selection of the terms to be subtracted, a uniformly valid closed-form approximation may be found.

In equation (F.17), the quantity to be matched is the last exponential factor, which can be written as

$$e^{-(\sqrt{\gamma}-1)k\xi\nu^2/(1+\nu^2-ik\nu)}=e^{-X\nu^2/[(\nu-ib)(\nu+i/b)]} \tag{F.19}$$

where

$$X=(\sqrt{\gamma}-1)k\xi \tag{101}$$

$$b=(k/2)+\sqrt{(k/2)^2+1} \tag{102}$$

An approximation of equation (F.19) has been found which yields correctly all of the results for the limiting cases previously discussed. This approximation is

$$e^{-X\nu^2/[(\nu-ib)(\nu+i/b)]}\simeq(1-e^{-X/b^2})e^{-X\nu^2}+(e^{-X/b^2}-e^{-X})e^{-X\nu^2/b^2}e^{-iX\nu/b}+e^{-X} \tag{F.20}$$

Equation (F.20) was arrived at by considering the form of the solution for large k. In this limit, the quantity b can be replaced by k. However, it can be shown that the approximation remains valid for small k if b is used instead of k. By adding and subtracting the terms on the right of equation (F.20) in the integrand of equation (F.17), the following results can be obtained

$$\frac{u(\tau,\xi)}{U}=\frac{u_A(\tau,\xi)}{U}+\frac{u_N(\tau,\xi)}{U}+0(\sqrt{\gamma}-1)^2 \quad \text{(F.21)}$$

where

$$\frac{u_A(\tau,\xi)}{U}=-\frac{i}{2\pi}\int_{-\infty}^{\infty}(e^{i\nu\tau}-e^{-i\nu|\tau|})e^{-i\nu\xi}[(1-e^{-X/b^2})e^{-X\nu^2}+(e^{-X/b^2}-e^{-X})e^{-X\nu^2/b^2}e^{-iX\nu/b}+e^{-X}]\frac{d\nu}{\nu} \quad \text{(F.22)}$$

and

$$\frac{u_N(\tau,\xi)}{U}=-\frac{i}{2\pi}\int_{-\infty}^{\infty}(e^{i\nu\tau}-e^{-i\nu|\tau|})\Big(e^{-i\nu\xi}\{e^{-X\nu^2/[(\nu-ib)(\nu+i/b)]}-(1-e^{-X/b^2})e^{-X\nu^2}+(e^{-X/b^2}-e^{-X})e^{-X\nu^2/b^2}e^{-iX\nu/b}+e^{-X}\}\frac{1}{\nu}-\frac{2(\sqrt{\gamma}-1)}{\gamma}\frac{(\sqrt{1-ik/\nu}-1)(\nu-i)}{(1+\nu^2-ik\nu)^2}(e^{c_2\nu\xi}-e^{-c_1\nu\xi})\Big)d\nu \quad \text{(F.23)}$$

In equation (F.21) the quantity $u_N(\tau,\xi)/U$ contains the remaining part of $u_c(\tau,\xi)/U$ not included in $u_A(\tau,\xi)/U$, and all of $u_r(\tau,\xi)/U$.

Equation (F.22) can be evaluated in closed form using reference 34 to obtain

$$\frac{u_A(\tau,\xi)}{U}=0 \qquad (\tau<0)$$

$$\frac{u_A(\tau,\xi)}{U}=\frac{1}{2}(1-e^{-X/b^2})\left[\operatorname{erf}\left(\frac{\tau-\xi}{2\sqrt{X}}\right)+\operatorname{erf}\left(\frac{\tau+\xi}{2\sqrt{X}}\right)\right]+\frac{1}{2}e^{-X}\left(1+\frac{\tau-\xi}{|\tau-\xi|}\right)+\frac{1}{2}(e^{-X/b^2}-e^{-X})\left\{\operatorname{erf}\left[\frac{b(\tau-\xi)-X}{2\sqrt{X}}\right]+\operatorname{erf}\left[\frac{b(\tau+\xi)+X}{2\sqrt{X}}\right]\right\} \qquad \tau>0 \quad (100)$$

In equation (F.23) the symmetry of the integrand can be used to reduce the interval of integration and eliminate the imaginary part of the integrand. Upon substitution of c_1 and c_2 from equations (F.1) and (F.2), the result can be written

$$\frac{u_N(\tau,\xi)}{U}=0 \qquad (\tau<0)$$

$$\frac{u_N(\tau,\xi)}{U}=\frac{2}{\pi}\int_0^{\infty}\frac{\sin\nu\tau}{\nu}\Big\{e^{-X\frac{\nu^2(1+\nu^2)}{(1+\nu^2)^2+k^2\nu^2}}\cos\left[\nu\xi+\frac{kX\nu^3}{(1+\nu^2)^2+k^2\nu^2}\right]-[(1-e^{-X/b^2})e^{-X\nu^2}+e^{-X}]\cos\nu\xi-(e^{-X/b^2}-e^{-X})e^{-X\nu^2/b^2}\cos\left[\nu\left(\xi+\frac{X}{b}\right)\right]+\frac{2(\sqrt{\gamma}-1)}{\gamma}\frac{\nu r_4 r_5}{r_6}[e^{-\delta_1\nu\xi}\cos(\lambda_1\nu\xi-\theta_7)-e^{-\delta_2\nu\xi}\cos(\lambda_2\nu\xi-\theta_7)]\Big\}d\nu \qquad (\tau>0) \quad \text{(F.24)}$$

where

$$\delta_1=\frac{(\sqrt{\gamma}-1)k\nu(1+\nu^2)}{(1+\nu^2)^2+k^2\nu^2} \quad \text{(F.25a)}$$

$$\lambda_1=1+(\sqrt{\gamma}-1)k^2\nu^2/[(1+\nu^2)^2+k^2\nu^2] \quad \text{(F.25b)}$$

$$\delta_2=\sqrt{\frac{\gamma+1}{2}}\cos\left(\frac{1}{2}\tan^{-1}\frac{k}{\nu}\right)\Big/[\nu(1+k^2\nu^2)^{1/4}] \quad \text{(F.25c)}$$

$$\lambda_2=\sqrt{\frac{\gamma+1}{2}}\sin\left(\frac{1}{2}\tan^{-1}\frac{k}{\nu}\right)\Big/[\nu(1+k^2\nu^2)^{1/4}] \quad \text{(F.25d)}$$

$$r_3=(1+k^2\nu^2)^{1/4} \quad \text{(F.25e)}$$

$$r_2=\sqrt{\frac{\gamma+1}{2}}\Big/\nu r_3 \quad \text{(F.25f)}$$

$$\theta_2=\frac{1}{2}\tan^{-1}\left(\frac{k}{\nu}\right) \quad \text{(F.25g)}$$

$$r_4=[(r_3\cos\theta_2-1)^2+r_3^2\sin^2\theta_2]^{1/2} \quad \text{(F.25h)}$$

$$\theta_4=\tan^{-1}[r_3\sin\theta_2/(1-r_3\cos\theta_2)] \quad \text{(F.25i)}$$

$$r_5=\sqrt{1+\nu^2} \quad \text{(F.25j)}$$

$$\theta_5=-\tan^{-1}(1/\nu) \quad \text{(F.25k)}$$

$$\theta_6=-2\tan^{-1}[k\nu/(1+\nu^2)] \quad \text{(F.25l)}$$

$$r_6=(1+\nu^2)^2+k^2\nu^2 \quad \text{(F.25m)}$$

$$\theta_7=\theta_4+\theta_5-\theta_6 \quad \text{(F.25n)}$$

Equation (F.24) is now in a form suitable for evaluation by machine computation as explained in appendix G.

APPENDIX G

NUMERICAL EVALUATION OF INTEGRALS

In this appendix, the results of the numerical evaluation of equation (F.24) by means of electronic machine computation will be discussed. Since the singularities have been removed, the only remaining difficulty in the numerical procedure is that associated with the infinite interval of integration. The difficulty cannot be removed by a transformation since there are a large number of zeros of the integrand in the interval. In the original integral, there were an infinite number of zeros of the integrand before subtraction of a function which was evaluated in closed form (eq. (100)). As a result it is only necessary to integrate equation (F.24) over some finite interval, beyond which the contribution is negligible. It is not feasible to set an arbitrary large interval of integration, because the required computing time is not negligible. Therefore the approximate required limit of integration was found as a function of the parameters k, γ, and the values of τ and ξ. To follow the variations of the integrand within this interval it was necessary to break the interval into one hundred parts. The calculations were then checked by doubling the interval of integration and doubling the number of points used. Since a large number of calculations were made, some effort toward optimizing the program was made. Finally a program was devised which would lead to valid evaluations of equation (F.24) for values of k between 0.001 and 1000 and γ between 1.0 and 2.0. A semiautomatic process for choosing the appropriate values of τ at which calculations should be made for given values of k, γ, and ξ was included.

For all values of the parameters and variables at which calculations were made, the part of the solution evaluated numerically (eq. (F.24)) was small compared to the total variation in $u(\tau, \xi)/U$ (equal to 1.0). Therefore the results from equation (100) were used in the text to summarize the findings from the numerical investigation. The results presented there are replotted in figures 6, 7, and 8 on an expanded scale for comparison with the more exact numerical computations. In figure 6 it can be seen that for small k the approximate dimensionless velocity u_A/U is correct to within 2 percent of the total transition. These results correspond to the data used in figure 4. The evaluations for intermediate k, used in figure 3, are compared with numerical values in figure 7. Here there is a maximum error in u_A/U of about 12 percent. This occurs at $\xi=1.819$ and $\tau=\xi$. The amplitude of the step at this point and at all other points is given correctly by u_A/U, but the approximation overestimates the magnitude of the precursor to the main part of the disturbance. Figure 8 shows the comparison for large k, corresponding to figure 5. The discrepancies are similar to those for intermediate k.

The results from the closed-form approximation u_A/U (eq. (100)) have been shown to be a good approximation. Therefore equation (100) could be used as a basis for a qualitative understanding of the effect of the neglected cumulative nonlinear terms which would be important at large distances from the wall. For this purpose, values of the temperature near $\tau=\xi$ and $\tau=\sqrt{\gamma}\xi$ are needed. These can be found from equation (100) using equations (2) and (4). The results for the temperature, found in this way, will not be correct in the entire flow field, but should be sufficiently accurate in the regions where they are needed for the nonlinear correction.

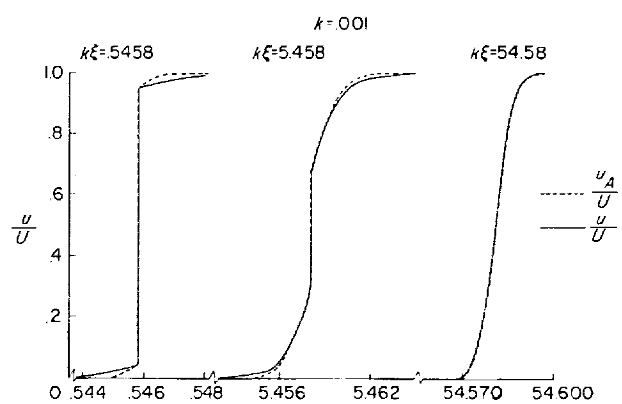

FIGURE 6.—Comparison of approximate and numerical evaluations of velocity response ($\kappa=0.001$).

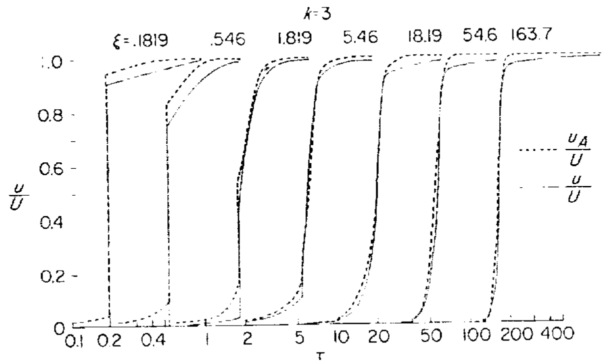

FIGURE 7.—Comparison of approximate and numerical evaluations of velocity response ($\kappa=3.0$).

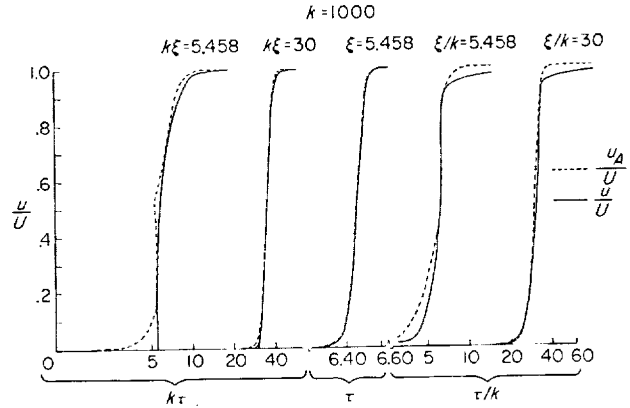

FIGURE 8.—Comparison of approximate and numerical evaluations of velocity response ($\kappa=1000$).

APPENDIX H

RESPONSE TO IMPULSIVE TEMPERATURE VARIATION OF A FIXED WALL

With boundary conditions

$$T'_w(t)=\begin{cases}0 & t<0\\ \Theta=\text{constant} & t>0\end{cases} \qquad \text{(H.1)}$$

$$u(t,0)=u_w(t)=0 \qquad \text{(H.2)}$$

and for an initial uniform state, equations (58) to (64) and (73) to (77) can be used to obtain the solution

$$u(\tau,\xi)=\frac{\Theta}{T_0}\frac{a_0}{\gamma}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{g_1-g_2}\right)(e^{c_1\nu\xi}-e^{c_2\nu\xi})e^{i\nu\tau}\frac{d\nu}{\nu} \qquad \text{(H.3)}$$

$$p'(\tau,\xi)=\frac{\Theta}{T_0}\frac{\rho_0 a_0^2}{\gamma}\left(\frac{-i}{2\pi}\right)\int_{-\infty}^{\infty}\left(\frac{1}{g_1-g_2}\right)\left(\frac{1}{c_1}e^{c_1\nu\xi}-\frac{1}{c_2}e^{c_2\nu\xi}\right)e^{i\nu\tau}\frac{d\nu}{\nu} \qquad \text{(H.4)}$$

$$\frac{T'(\tau,\xi)}{\Theta}=-\frac{i}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{g_1-g_2}\right)\left(\frac{1+\frac{c_1^2}{\gamma}}{c_1}e^{c_1\nu\xi}-\frac{1+\frac{c_2^2}{\gamma}}{c_2}e^{c_2\nu\xi}\right)(e^{i\nu\tau}-1)\frac{d\nu}{\nu} \qquad \text{(H.5)}$$

where

$$g_1=\frac{1+\frac{c_1^2}{\gamma}}{\left(1+\sqrt{\frac{2}{\gamma+1}}\,\nu c_1\right)c_1},\qquad g_2=\frac{1+\frac{c_2^2}{\gamma}}{\left(1+\sqrt{\frac{2}{\gamma+1}}\,\nu c_2\right)c_2} \qquad \text{(H.6)}$$

The variables and parameters have the same definitions here as in the previous problem (see the table of symbols in appendix A).

Using the approximation scheme discussed in appendix F, the integrals can be simplified to the following forms

$$u(\tau,\xi)=\frac{\Theta}{T_0}a_0\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\sqrt{\nu(\nu-ik)}-\nu}{\nu^2-ik\nu+1}\right](e^{c_1\nu\xi}-e^{c_2\nu\xi})(e^{i\nu\tau}-e^{-i\nu|\tau|})\frac{d\nu}{\nu}+0(\sqrt{\gamma}-1) \qquad \text{(H.7)}$$

$$p'(\tau,\xi)=\frac{\Theta}{T_0}\rho_0 a_0^2\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\sqrt{\nu(\nu-ik)}-\nu}{\nu^2-ik\nu+1}\right][e^{c_1\nu\xi}-i\sqrt{\nu(\nu-ik)}e^{c_2\nu\xi}](e^{i\nu\tau}-e^{-i\nu|\tau|})\frac{d\nu}{\nu}+0(\sqrt{\gamma}-1) \qquad \text{(H.8)}$$

$$\frac{T'(\tau,\xi)}{\Theta}=\frac{-i}{2\pi}\int_{-\infty}^{\infty}\left(1-\sqrt{\frac{\nu}{\nu-ik}}\right)e^{-\xi\sqrt{\frac{\nu}{\nu-ik}}}(e^{i\nu\tau}-1)\frac{d\nu}{\nu}+0(\sqrt{\gamma}-1) \qquad \text{(H.9)}$$

where

$$c_1=-i+(\sqrt{\gamma}-1)\frac{ik\nu}{\nu^2-ik\nu+1}+0(\sqrt{\gamma}-1)^2 \qquad \text{(H.10)}$$

$$c_2=-\frac{1}{\sqrt{\nu(\nu-ik)}}+0(\sqrt{\gamma}-1) \qquad \text{(H.11)}$$

Equation (H.9) can be evaluated approximately, in a form that can be shown to be correct for all limiting values of the variables and parameters. The result is

$$\frac{T'(\tau,\xi)}{\Theta}\approx(1-e^{-\xi})\left[1-\operatorname{erf}\left(\frac{\xi}{2\sqrt{k\tau}}\right)\right]+e^{-\xi}\left(1-e^{-\frac{k\tau}{2}}\right) \qquad \text{(H.12)}$$

Equation (H.12) indicates that no discontinuities in temperature develop. It can be verified from equation (H.9) and also from equation (H.5) that there is no discontinuity in temperature at $\tau=\xi$, where one might be expected. Also the following qualitative behavior of the gas temperature can be deduced from equation (H.12). At a point near the wall (small ξ) the last term dominates. This function indicates a rise in perturbation temperature from zero (at $\tau=0$) to a value equal to the wall temperature, in a time when τ becomes of order k^{-1}. This variation includes a nonzero initial slope, which can be verified in the exact equation (H.5). At a point far from the wall ($\xi>>1.0$), the first term in equation (H.12) dominates. This function also indicates a rise in perturbation temperature from zero at $\tau=0$ to a value equal to the wall temperature at a later time. But here the initial slope is zero, and the variation occurs in a region near $\tau=k\xi^2$. Thus, at large distance from the wall, the temperature variation assumes the character of a diffusion process.

No closed-form approximation for the velocity has been found. However, the following properties of the solution can be deduced from equation (H.7). The velocity disturbance is everywhere small compared to that associated with the response to an impulsive wall motion involving comparable temperature changes. The velocity goes to zero at all limiting values of the parameters and variables. No discontinuities in velocity develop at $\tau=\xi$. The last two findings can be verified in the exact equation (H.3). At a point near the wall (small ξ), the velocity rises with a nonzero initial slope, but eventually returns to zero. Exactly at the wall, the velocity is zero at all times according to the boundary condition for a fixed wall (eq. (H.2)). At a point somewhat removed from the wall (intermediate ξ), the velocity disturbance consists of two parts. One of these is associated with the temperature disturbance and, hence, leads to a rise in velocity near $\tau=k\xi^2$. The velocity returns to zero at values of τ large compared to $k\xi^2$. Also, as ξ approaches infinity, this component of the velocity disturbance goes to zero for all values of τ. There is a second part of the velocity disturbance at intermediate distance from the wall. This component has a peak near $\tau=\xi$ and hence represents a compression wave resulting from the nonuniform heating. This wave travels at a speed between the isothermal and isentropic signal speeds. As it progresses, it builds up initially, but subsequently decays to zero at large distances from the wall. This part of the disturbance also goes to zero at large τ for all values of ξ. There is a small variation in gas temperature associated with this compression wave, but it is of order $(\sqrt{\gamma}-1)$ and is not included in the approximate expression for the temperature given in equation (H.12). For $k=0$ or $k\rightarrow\infty$ the total velocity disturbance goes to zero everywhere.

Since the velocity disturbance is small and goes to zero at $\xi\rightarrow\infty$ in the present problem, the cumulative nonlinear effects will be negligible for larger disturbances than they would be for the impulsive-motion case. Also from a mathematical point of view, it is interesting to note that, for the present impulsive wall-temperature problem, the linear approximation is uniformly valid in the limit of a vanishingly small disturbance. In this limit, no discontinuities of the shock-wave type develop, although discontinuities in the derivatives of the flow quantities do occur at $\tau=0$ and at $\tau=\xi$.

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